BACKGROUND INFO SHEET II

This is again just for reference

Basic notions for the homotopy theory of graphs.

In what follows we think of a graph as its geometric realization. Thus it is in particular a topological space, but vertices and edges are still part of its structure. We will always think of a graph as being oriented. We parametrize each edge e by $[0,1]$, i.e. we have a continuous map $f_e : [0,1] \to e \subset \Gamma$ which is an orientation preserving embedding when restricted to $(0,1)$ and maps 0 to $\iota(e)$ and 1 to $\tau(e)$.

1. Proposition. Let Γ be a connected graph, $x_0 \in \Gamma$ (x_0 need not be a vertex) and Y a topological space and $y_0 \in Y$.

(a) To any homomorphism $\varphi : \pi_1(\Gamma, x_0) \to \pi_1(Y, y_0)$ there exists a continuous map $f : (\Gamma, x_0) \to (Y, y_0)$ inducing φ

(b) Let $f_0, f_1 : (\Gamma, x_0) \to (Y, y_0)$ be two cont. maps inducing the same homom. on π_1. Then there exists a homotopy $f : \Gamma \times I \to Y$ from f_0 to f_1 with $f(\{x_0\} \times I) = y_0$

Proof: (a) should be obvious. First, if x_0 is not a vertex, subdivide the edge containing x_0 into two edges so that x_0 becomes a vertex. This does not change Γ as a topological space.

Now pick a maximal tree T in Γ. For each edge e not in T let $w_e = [x_0, \iota(e)]_T \, e \, [\tau(e), x_0]_T$, considered as a loop in Γ. Then $\{[w_e] : e \text{ not in } T\}$ is a basis of the free gp. $\pi_1(\Gamma, x_0)$. Let $v_e : [0,1] \to Y$ be a representing loop of $\varphi([w_e]) \in \pi_1(Y, y_0)$. Then define $f : (\Gamma, x_0) \to (Y, y_0)$ as follows:

$f(T) = \{y_0\}$. $\quad f|_e = v_e \quad$ i.e.

if $f_e : [0,1] \to e$ parametrizes e then

$$f(f_e(t)) = v_e(t), \quad 0 \leq t \leq 1.$$

Then $f_*([w_e]) = [v_e] = \varphi([w_e])$. Since $\{[w_e] : e \text{ not in } T\}$ is a basis $f_* = \varphi$.

(b) Not much harder than the proof of (a). Again we may assume that $x_0 \in V_\Gamma$. Then

$$\Gamma \times I = (V_\Gamma \times I) \cup (\text{edges in } T) \times I \cup (\text{edges not in } T) \times I.$$

We construct the homotopy f from f_0 to f_1 starting on $V_\Gamma \times I = \{x_0\} \times I \cup (V_\Gamma \setminus \{x_0\}) \times I$

mapping $\{x_0\} \times I$ to y_0 and for $v \in V_\Gamma \setminus \{x_0\}$ we map $v \times I$ to the path

$$f_0 \circ [v, x_0]_T * f_1 \circ [x_0, v]_T \quad \text{i.e}$$

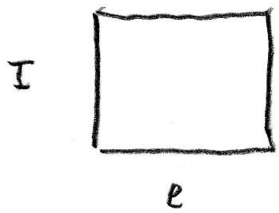
if $w_v : [0,1] \to T$ parametrizes the path $[x_0, v]_T$

then $(v,t)\in v\times T$ gets mapped to (by f)

$$\begin{cases} f_0(w_v(1-2t)) & 0\le t\le \frac{1}{2} \\ f_1(w_v(2t-1)) & \frac{1}{2}\le t\le 1 \end{cases}$$

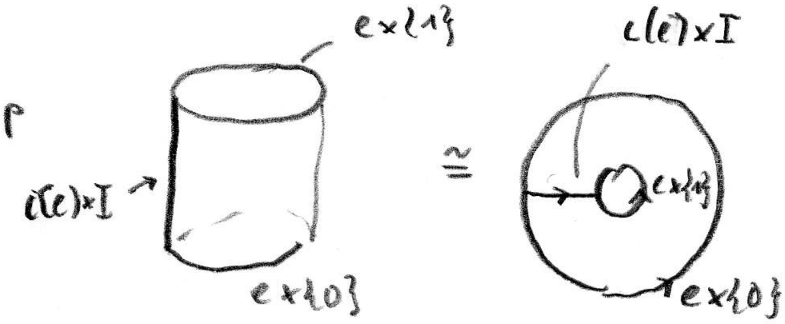
In particular, $f(v,0)=f_0(v)$, $f(v,\frac{1}{2})=x_0$, $f(v,1)=f_1(v)$.

Now let e be an edge of T; look at $e\times I$



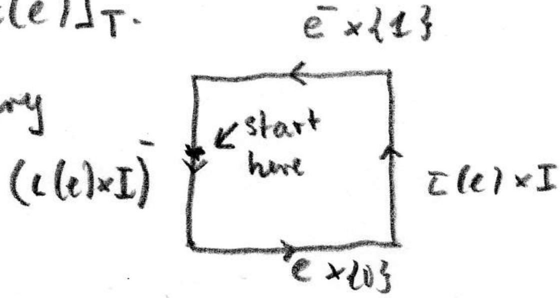
then f is already defined on the boundary of the square or

in case of a loop



The complement is in both cases the interior of a square. For simplicity assume that e is oriented in such a way that e is part of $[x_0,\tau(e)]_T$ (and thus e is not part of $[x_0,\iota(e)]_T$.

Then the boundary



of the square is mapped to

$$f_0\circ[x_0,\iota(e)]_T * f_0(e) * f_0\circ[\tau(e),x_0]_T * f_1\circ[x_0,\tau(e)]_T \circ f_1(e)^{-} * f_1[\iota(e),x_0]_T$$

Now the product of the first three factors is

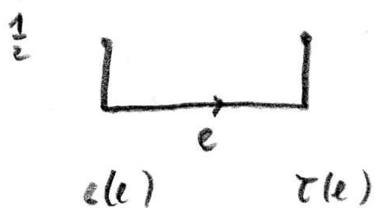
$$f_0\Big(\underbrace{[x_0, \iota(e)]_T \, e \, [\tau(e), x_0]_T}_{= [x_0, \tau(e)]_T \, [\tau(e), x_0]_T} \Big)$$

, i.e. it is the image under f_0 of the path in T from x_0 to $\tau(e)$ and back.

So it is homotopic as a path to the constant path c_{x_0}.
The same holds for the product of the last three factors.
Therefore, we can extend the map f to all of $e \times I$.

It remains to extend f to $e \times I$ for edges e not in T. Finally, we have to make use of the hypotheses, that f_0 and f_1 induce the same map in π_1.

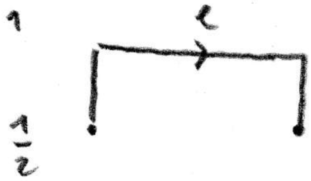
Again, we have to check that the map given on the boundary of $e \times I$ (where in the case of a loop edge, we first cut the cylinder $e \times I$ along $\iota(e) \times I$ to get a square) is homotopic to a constant map.



$$\xrightarrow{f} \quad f_0 \circ [x_0, \iota(e)]_T * f_0 \circ e * f_0 [\tau(e), x_0]_T$$

$= f_0 \circ w_e$ which represents $f_{0*}([w_e])$

Similarly



gets mapped by f to

$f_1 \circ w_e$ which represents $f_{1*}([w_e])$

Since $f_{1*} = f_{0*}$ $f_0 \circ w_e$ and $f_1 \circ w_e$ are homotopic as paths. In particular $f_0 \circ w_e * (f_1 \circ w_e)^{-}$ is homotopic to a constant path and we can extend. $\square$

2. <u>Corollary</u>: Let $f : (\Gamma, x_0) \longrightarrow (\Gamma', x_0')$ be a map between graphs inducing an isomorphism on π_1. Then f is a homotopy equivalence.

<u>Pf.</u> Let $g : (\Gamma', x_0') \longrightarrow (\Gamma, x_0)$ be a map inducing $(f_*)^{-1}$, i.e. $g_* = (f_*)^{-1}$. Then $g \circ f$ and id_Γ induce the same map on $\pi_1(\Gamma, x_0)$. Thus $g \circ f \simeq id_\Gamma$. Similarly $f \circ g \simeq id_{\Gamma'}$. $\square$

3. Let $f : (\Gamma, x_0) \to (\Gamma, x_0)$ induce an inner automorphism on $\pi_1(\Gamma, x_0)$, where Γ is a connected graph. Then f is homotopic to id_Γ.

<u>Remark</u>: While in 1 and 2 we could always choose our homotopies to preserve basepoints this no longer holds in 3.

Pf. (See lecture on Jan. 21)

4. Final Remark. Let X be path-connected, $x_0 \in X$.

$G = \pi_1(X, x_0)$. Let $[X, X]^*$ be the set of homotopy classes of homotopy equivalences $f : X \to X$. Define a multiplication on $[X, X]^*$ by $[f] \cdot [g] = [f \circ g]$. Then $[X, X]^*$ is a group and there is a well-defined homomorphism $[X, X]^* \longrightarrow \mathrm{Out}\, G$ given by

$$[f] \longrightarrow \{h_w \circ f_*\} \quad \text{where}$$

$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(X, f(x_0))$ is the isomorphism induced by f, w is a path from x_0 to $f(x_0)$ and

$h_w : \pi_1(X, f(x_0)) \longrightarrow \pi_1(X, x_0)$ is the isomorphism given by $[\alpha] \longmapsto [w * \alpha * w^{-1}]$, α a loop in X beginning in $f(x_0)$. $\{\ \}$ means outer autom. class.

$[X, X]^* \longrightarrow \mathrm{Out}\, G$ is injective if every homotopy equivalence $f : (X, x_0) \to (X, x_0)$ which induces an inner automorphism is homotopic to id. It is surjective if for any automorphism φ of G there is a homotopy equivalence $f : (X, x_0) \to (X, x_0)$ inducing φ.

[By 1. and 2. these conditions are satisfied if X is a connected graph.] i.e.

[If Γ is a graph $[\Gamma, \Gamma]^* \cong \mathrm{Out}(\pi_1(\Gamma, x_0))$]

Proof of final remark.

That $[f]\cdot[g] := [f\circ g]$ makes $[X,X]^*$ into a group is obvious. Neutral element is $[id_X]$ and $[f]^{-1} = [g]$, if g is any htpy inverse of f. Associativity of „$\cdot$" is obvious.

We know that $f \longmapsto \{h_w \circ f_*\}$ does not depend on the path w from x_0 to $f(x_0)$.

If f' is homotopic to f, let $F: X\times I \longrightarrow X$ be a homotopy from f to f', i.e. $F(x,0) = f(x)$, $F(x,1) = f'(x)$. Let $v(t) = F(x_0, t)$. Then v is a path from $f(x_0)$ to $f'(x_0)$.

We claim that $h_{w*v} \circ f'_* = h_w \circ f_*$. To see this, notice that $h_{w*v} = h_w \circ h_v$. Thus it suffices to show that $h_v \circ f'_* = f_*$. To see this let $F_t : X \to X$ be given by $F_t(x) = F(x,t)$ and for $0 \leq t \leq 1$ let $v_t : [0,1] \longrightarrow X$ be given by

$v_t(s) = v(t\cdot s)$, a path from $f(x_0)$ to $v(t)$.

If α is any loop based at x_0 then

$f_*[\alpha] = [f\circ\alpha] = [F_0 \circ \alpha]$ for $t \in [0,1]$

Let α_t be the path $v_t * F_t \circ \alpha * v_t^-$ then

$[\alpha_0] = [F_0 \circ \alpha]$, $[\alpha_1] = [v * F_1 * v^-] = [v * f' \circ \alpha * v^-]$

$[F_0\circ\alpha] = f_*([\alpha])$, $\quad [v * f'\circ\alpha * v^-] = h_v \circ f'_*[\alpha]$

This shows that

$$[f] \longmapsto \{h_w \circ f_*\} \text{ is a well defined map}$$

$$[X,X]^* \longrightarrow \text{Out } G.$$

Let f, g be homotopy equivalences. Let v be a path from x_0 to $g(x_0)$. Then $f \circ v$ is a path from $f(x_0)$ to $fg(x_0)$.

Thus $[f \circ g] \longmapsto \{h_{w * f\circ v} \circ (f \circ g)_*\}$ if w is any path from x_0 to $f(x_0)$. We have to compare this to

$$[f]\cdot[g] \longmapsto \{(h_w \circ f_*) \circ (h_v \circ g_*)\}$$

So let α be a loop based at x_0. Then

$h_v \circ g_* [\alpha] = [v * g\circ\alpha * v^-]$. Applying f_* to this we get $[(f\circ v) * f\circ g\circ\alpha * (f\circ v)^-]$. Applying h_w to this we get $[w * f\circ v * f\circ g\,\alpha * (w * f\circ v)^-]$

$$= h_{w * f\circ v} \circ (f\circ g)_* [\alpha]$$

$\square$