Covering some of the missing steps in our proof that Out_n, $n>1$, is contractible.

6.A.1. A (hopefully complete) proof of 6.2.4

$(Out_n; \text{weak}) \xrightarrow{id} (Out_n; G\text{-}H)$ is a homeomorphism.

"$\rightarrow$" is continuous is easier. Let T be a free metric F_n-tree as defined in 6.1, i.e. F_n acts freely by isometries, T/F_n is a finite graph, the valence of each vertex of T is at least 3. We look at non-normalized outer space, i.e. $\mathrm{vol}(T/F_n) \in \mathbb{R}_{>0}$; the weak topology one gets by taking our original Out_n and multiply by $\mathbb{R}_{>0}$. Call the corresponding space $\mathcal{D}_n$ (i.e. $\mathcal{D}_n = Out_n \times \mathbb{R}_{>0}$).

The Gromov-Hausdorff-Topology G-H is defined as before. $\mathcal{D}_n$ is in our earlier notation the set of strong equivalence classes of free metric F_n-trees.

To prove, that $\xrightarrow{id}$ is continuous, we have to provide for every $V_{x_1,\dots,x_r,g_1,\dots,g_s,\varepsilon}(T)$ a neighborhood of T in the weak topology contained in $V_{x_1,\dots,x_r,g_1,\dots,g_s,\varepsilon}(T)$

Let $m_T = \min\{\ell(e) : e \text{ edge of } T\}$. Here $\ell(e)$ is the length of e. Since T/F_n is finite, $\ell(e)$ is the length of the shortest edge of T/F_n.

Let $0 < \delta < m_T$. Let $N_\delta(T)$ be the set of metric trees obtained from T as follows.

Replace each vertex v of T/F_n by a metric tree T_v of volume $\leq \delta$. For each edge of T/F_n having v in its boundary choose a vertex of T_v to attach the edge. If the edge is a loop you are allowed to attach intial and end point to different vertices of T_v.

But do this in such a way that the new graph Γ' is legal (i.e. all vertices have valence at most 3) This implies that T_v has at most $k-3$ edges if v has valence k.

Now finally, you are allowed to change the lengths of edges of Γ' coming from edges of T/F_n by at most δ.

Let T' be the universal cover of Γ' (We know that any two universal covers of Γ' with the metrics induced from Γ' are strongly equivalent; so we may speak of <u>the</u> universal cover

Let $N_\delta(T)$ be the set of these T' in $\mathcal{D}_n$.

(Since the Euler characteristic of a tree is the same as the Euler characteristic of a point we have

$$\mathrm{char}\,(T/F_n) = \mathrm{char}\,(\Gamma')\ ,\ \text{so that } \pi_1(\Gamma') \cong F_n.$$

Thus each T' becomes a free metric F_n-tree)

You should have no problem to show that $N_\delta(T)$ is a nbhd. of T in $\mathcal{D}_n$ in the weak topology.

<u>Claim</u>: For some $\delta > 0$. ($\delta \leq m_T$) we have $N_\delta(T) \subset$

$$V_{x_1,\dots,x_r;\, g_1,\dots,g_s,\, \varepsilon}(T).$$

Given $T' \in N_\delta(T)$ there is a more or less obvious choice for picking $\tilde{x}_1, \dots, \tilde{x}_r$.

First: for each $\overbrace{\text{edge}}^{\text{oriented}}$ $e \in T$ there is a corresponding edge e' of T' and $|\ell(e) - \ell(e')| \leq \delta$. Further for each vertex $v \in T$ there is a corresponding tree T_v in T' and $\mathrm{vol}(T_v) \leq \delta$.

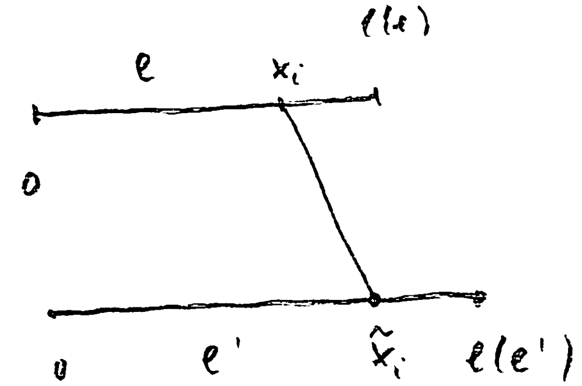
If $x_i \in$ interior e pick $\tilde{x}_i \in$ interior e' such that x_i and $\tilde{x}_i$ have the same parameter in the corresponding $\overbrace{\text{metric}}^{\text{normalized}}$ parametrization:

If $[0, \ell(e)] \xrightarrow{\varphi} e$ parametrizes e and

$x_i = \varphi(t \cdot \ell(e))$ then $\tilde{x}_i = \varphi'(t \cdot \ell(e'))$.

e

so



Since $|l(e') - l(e)| \leq \delta$ the distances $d_T(x_i, a)$ and $d_{T'}(\hat{x}_i, \hat{a}')$ differ by at most δ, where a is one of the endpoints of e, and a' the corresp. endpoint.

If x_i is a vertex, pick any $\tilde{x}_i$ (vertex) in T_{x_i}.
Again: if e is an edge in T with endpoint x_i and $x \in e$, $x' \in e'$ the corresponding point.
Then $d_T(x, x_i)$ and $d(x', \tilde{x}_i)$ differ by at most 2δ.

Now look at a tripel x_i, x_j, g_k.
look at $x_i, g_k(x_j)$ and let $[x_i, g_k(x_j)]_T$ run through α edges and β vertices of T where at the beginning and end we might run through a part of an edge (in any case $|\alpha - \beta| \leq 1$) then $[\tilde{x}_i, g_k(\tilde{x}_j)]_{T'}$ differs in length from

$[x_i, g_k(x_k)]_T$ by at most $(\alpha+\beta)\cdot\delta$.

Let K be the maximum $(\alpha+\beta)$ occurring, when we run through all tripels (x_i, x_j, g_k). Now choose $\delta < \frac{\varepsilon}{K}$ (and $\delta < m_T$). $\square$

$\boxed{\text{End of proof that } \xrightarrow{id} \text{ is continuous}}$

Proof that $\xleftarrow{id}$ is continuous.

Given $N_\delta(T)$, we need to find $x_1, \ldots, x_r, g_1, \ldots, g_s, \varepsilon$ with $V_{x_1,\ldots,\varepsilon} \subset N_\delta(T)$.

We first choose a tree $\overline{T_0}$ of T/F_n. We fix some point $x_1 \in T$ and let $\overline{x_1}$ be its image in T/F_n. Choose an orientation for the edges of T/F_n not in $\overline{T_0}$ and label them $\overline{e}_1, \ldots, \overline{e}_n$.

The inverse image of $\overline{T_0}$ in T consists of disjoint copies of $\overline{T_0}$. The copy containing x_1 is denoted by T_0. The inverse image of $\overline{T_0}$ in T is then $\coprod_{g\in F_n} g(T_0)$.

Denote the lifts of $\overline{e}_i$ to T with initial points in T_0 by $e_1, \ldots, e_n$.

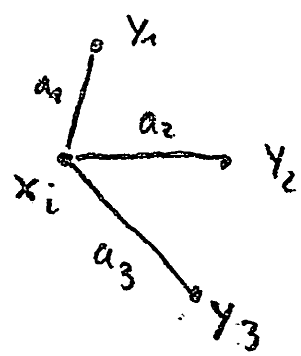
Then there exist unique $h_i \in F_n$, $i=1,\dots,n$, such that $h_i^{-1}(\tau(e_i)) \in T_0$. Let $x_1, \dots, x_r$ be the vertices of T_0, let $\{g_1, g_2, \dots, g_{2n+1}\} = \{1, h_1, h_1^{-1}, \dots, h_n, h_n^{-1}\}$. We claim that there exists $\varepsilon > 0$ s.t.

$$V_{x_1,\dots,x_r;\, g_1,\dots,g_{2n+1};\, \varepsilon}(T) \subset N_\delta(T).$$

As a first approximation to our final choice of ε we make $\varepsilon < \frac{1}{6} m_T$.

For each x_i, $1 \le i \le r$, we find in

$X := \{ g_k(x_j) : 1 \le j \le r,\ 1 \le k \le 2n+1 \}$ three distinct vertices y_1, y_2, y_3 with $x_i \notin \{y_1, y_2, y_3\}$ such that y_k and x_i are joined by an edge a_k in T, $k = 1, 2, 3$.



Let $T' \in V_{x_1,\dots,g_{2n+1},\varepsilon}(T)$.

and $\tilde{x}_1, \dots, \tilde{x}_r \in T'$ the points satisfying the defining inequalities for $V_{x_1, \dots, g_{2n+1}, \varepsilon}(T) =: V_\varepsilon(T)$.

Let $\tilde{y}_k$, $k = 1, 2, 3$, be the points in

$$\{ g_\ell(\tilde{x}_j) : 1 \le j \le r,\ 1 \le \ell \le 2n+1 \} \text{ corresponding}$$

to y_k.

Then $d(y_\ell, y_k) = \ell(a_\ell) + \ell(a_k)$ if $\ell \neq k$.

$$d(y_\ell, x_i) = \ell(a_\ell), \quad \ell = 1, 2, 3.$$

If $d(\tilde{y}_\ell, \tilde{y}_k) = d(\tilde{y}_\ell, \tilde{x}_i) + d(\tilde{y}_k, \tilde{x}_i)$ then

$\tilde{x}_i$ must be a vertex of T'.

If equality does not hold there must be backtracking in $[\tilde{y}_\ell, \tilde{x}_i]_{T'}$, $[\tilde{x}_i, \tilde{y}_k]_{T'}$ taking place at $\tilde{x}_i$ for some pair k, ℓ, $k \neq \ell$, say $k = 1$, $\ell = 2$

Now $|d(\tilde{y}_j, \tilde{x}_i) - \ell(a_j)| \le \varepsilon$ $\quad j = 1, 2$

$$|d(\tilde{y}_1, \tilde{y}_2) - (\ell(a_1) + \ell(a_2))| \le \varepsilon$$

Thus $d(\tilde{y}_1, \tilde{y}_2) \ge d(\tilde{y}_1, \tilde{x}_i) + d(\tilde{x}_i, \tilde{y}_2) - 3\varepsilon$

Consequently, there is a vertex of T' in the $\frac{3}{2}\varepsilon$-neighborhood of $\hat{x}_i$. Since $6\varepsilon < m_T$ $(= \min\{\ell(a) \mid a \text{ edge of } T\}$ and since $d(\tilde{y}_j, \tilde{x}_i) \geq \ell(a_j) - \varepsilon$ backtracking stops $\frac{\varepsilon}{2}$ before the midpoint of $[\tilde{y}_j, \hat{x}_i]_{T'}$.

For each i let T_i be the convex hull of all vertices of T' in the $\frac{3}{2}\varepsilon$ neighbourhood of $\hat{x}_i$. Then clearly, the diameter of T_i is $\leq 3\varepsilon$. Thus, if s is the number of edges of T_i we have $\mathrm{vol}(T_i) \leq 3s\cdot\varepsilon$. Let now x_i, x_j be vertices of T_0 connected by an edge a. Since $\hat{x}_i \in T_i$, $\hat{x}_j \in T_j$ and $|d(\hat{x}_i, \hat{x}_j) - \ell(a)| < \varepsilon$ and points of T_i are at most $\frac{3}{2}\varepsilon$ from $\hat{x}_i$ (and similarly for j) there is a path of length at least ε and at most $\ell(a) + 4\varepsilon$ in T connecting T_i and T_j. We call this the path corresponding to a. Similarly, if x_i is a vertex of T_0 and $Y = g_k(x_j)$ a vertex of X connected to x_i by an edge a, we find a corresponding edge in T' from $g_k(T_j)$ to T_i.

Thus, if p_i is the valence of x_i in T there are p_i paths in T' in the complement of T_i and ending in T_i. Consequently, T_i has at most $p_i - 3$ edges. Since for any free metric F_n tree the valence of every vertex is at most $2n$ we demand that $\varepsilon \leq \dfrac{\delta}{3 \cdot (2n-3)}$.

(we have already assumed that $\varepsilon < \frac{1}{6} m_T$)
Since $n \geq 2$ we have in particular, that $\varepsilon \leq \frac{1}{3}\delta$.

Let us review what we know about T'.

- For each vertex of T_0 a tree T_i in T' of volume $\leq \delta$. Each T_i contains $\tilde{x}_i$ and $d(y, \tilde{x}_i) < \frac{3}{2}\varepsilon$ for all $y \in T_i$.

- For any other vertex v in T there exists a unique i, $1 \leq i \leq r$, and $g \in F_n$ s.t. $v = g(x_i)$. We set $T_v = g(T_i)$.

- If $x_i \in T_0$, $y = g_k(x_j)$ in X and if there is an edge in T from x_i to y then there is a reduced path in the complement of T_i and

T_y in T' connecting T_i and T_y. ($T_i = T_{x_i}$).

6.A.1.1 <u>We claim</u>:

(i) Every vertex of T' is a vertex in some T_v, v a vertex of T (set $T_{x_i} = T_i$).

(ii) for every edge a' in T' not in the union of the T_v, v a vertex of T, there is $x_i \in T_0$,

$$y \in \{ g_k(x_j) : 1 \le j \le r \; ; \; g_k \in \{1, h_1, \dots, h_n\} \}$$

such that x_i is connected to y either by an edge in T_0 if $g_k = 1$, or by e_s if $g_k = h_s$.

and a $g \in F_n$ such that $a' = g(\tilde{a})$.

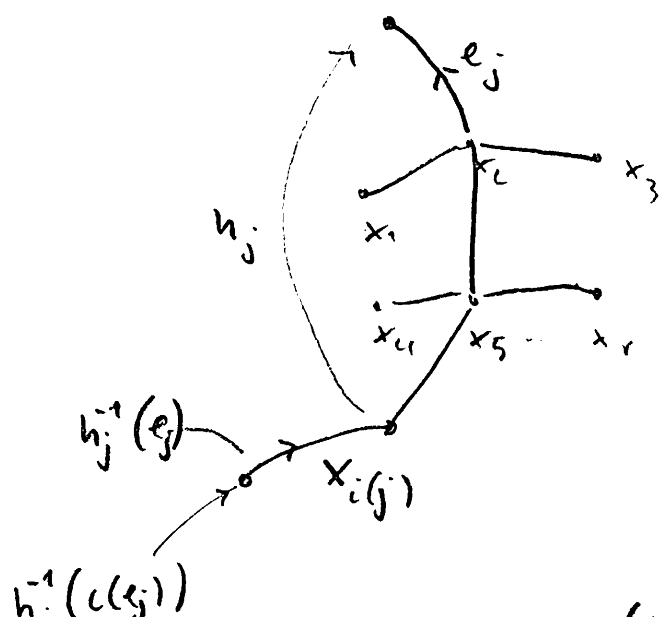
Here $\tilde{a}$ is the edge from T_{x_i} to T_y associated to the edge from x_i to y.

Once we have proved 6.A.1.1 the proof of 6.A.1 is complete.

We first concentrate on the part of T' directly involved with $\{\tilde{x}_i : 1 \le i \le r\}$.

We have the trees $T_{x_i} = T_i$ $1 \le i \le r$. By our choice of ε they are pairwise disjoint.

For each $j = 1, \dots, n$, we have the edges e_j, $j = 1, \dots, n$. $\iota(e_j) \in T_0$ and for each j we have the unique vertex $x_{i(j)} \in T_0$ with $h_j(x_{i(j)}) = \tau(e_j)$



Then $T_{\tau(e_j)} = h_j(T_{i(j)})$ by definition, and

$T_{\tau(e_j)} \cap T_{x_i} = \emptyset$, if $x_i = \iota(e_j) \in T_0$.

Consider $\{T_{x_i} : 1 \le i \le r\} \cup \{T_{\tau(e_j)} : j = 1, \dots, n\}$.

These trees in T' are pairwise disjoint by construction.

Let $a_1, \dots, a_{r-1}$ be the edges of T_0, and $\tilde{a}_1, \dots, \tilde{a}_{r-1}$ the associated paths in T'! Similarly $\tilde{e}_1, \dots, \tilde{e}_n$