Points to consider:

Our construction depended only on one choice: the choice of the "base" point $P \in T$ $(= T_\infty)$. After this choice, everything we did was dictated by the geometry.

- We need a continuous choice of P to have a chance of making our contraction continuous.
- We should convince ourselves that each path is continuous.

These two points we will deal with today. And this will end the official part of the lecture.

We will expose some ideas of the continuity of the contraction next week for those who are willing to sacrifice some time of the semester break.

Continuous choice of basepoint.

Given a free metric F_n tree T (as defined in 6.1.11 (a)). We want to associate to T a point $P = P_T$ such that

(i) If $\varphi : T \to T'$ is an equivariant homothety then $\varphi(P_T) = P_{T'}$.

(ii) If $T_i \to T$ converges to T in the weak topology and $\mathrm{vol}(T_i/F_n) = \mathrm{vol}(T/F_n) = 1$ for all i then $P_{T_i} \to P_T$

Remark. To give condition (ii) some meaning, recall

A neighborhood of T can be described as follows. (Here we do not assume that $\mathrm{vol}(T/F_n) = 1$. If necessary we can scale down. So we look at <u>strong</u> equivalence for now)

Let $\ell = \min$ (length of edges of T)
Since T/F_n is finite we have $\ell > 0$.

Now choose $\varepsilon < 0$; $\varepsilon < \frac{\ell}{2}$.

for every orbit $F_n \cdot v$, v vertex of T,

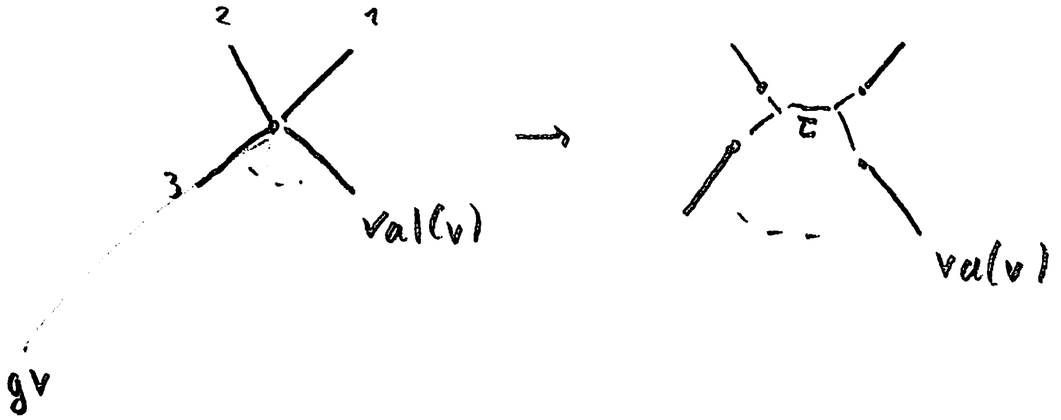
consider the set

$$\mathcal{T}_v = \{\text{finite metric tree of volume} < \varepsilon \text{ with } \mathrm{val}(v) \text{ leaves and every non-leaf vertex having valence} \geq 3\}$$

Then consider

$N_\varepsilon(T)$ the set of metric trees where

$T' \in N_\varepsilon(T)$ if T' is obtained from T by replacing each orbit $F_n \cdot v$ of vertices of T by picking an element $\tau \in \mathcal{T}_v$ inserting in each $g \cdot v$ a copy of τ as follows



→ remove vertices of valence 2.

We also allow ~~before~~ this process to change the lengths of orbits of edges of T by at most ε.

(So edges corresponding to "old" edges in T can get shorter by less than ε and longer by less than 3ε)

So if we normalize volume we might have to "stretch" (or shrink) T' by at most 2·α if T has to be stretched by α.

Thus $N_\varepsilon(T)$ maps down to a neighborhood of normalized T in the weak topology.

The action of F_n on T' extends the action of T in the obvious way. The copy of τ at v is mapped via "identity" to the copy of τ at gv. Other edges of T' correspond to edges of T and are mapped as before, but isometrically.

In this neighborhood, there is a well defined map $T' \xrightarrow[f_{T'}]{} T$ for every $T' \in N_\varepsilon(T)$; collapse the trees and adjust lengths of remaining edges.

We call or choices of $P_{T'}$ continuous if for every $T_i' \longrightarrow T$, $T_i' \in N_\varepsilon(T)$ we have

$$f_{T_i'}(P_{T_i'}) \longrightarrow P_T \quad \text{in } T.$$

Choice of basepoint according to Guirardel + Levitt

We need one more concept.

Let T be a free metric F_n-tree. For any $g \in F_n$ consider

$$A_g := \{ x \in T : d(x, g(x)) = \inf_{y \in T} \{ d(y, g(y)) \} \}$$

6.3.1 Claim: $A_g \subset T$ is isometric to $\mathbb{R}$. It is called the axis of g.

Proof. Let $y \in T$. There exists a unique path $a := [y, g(y)]_T$ in T from y to $g(y)$. Then $g(a)$ is a path from $g(y)$ to $g^2(y)$.

There are two possibilities:

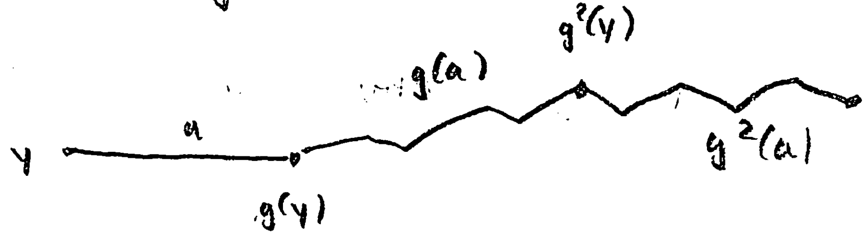
$$g(a) \wedge a \begin{array}{l} = g(y) \\ \neq g(y). \end{array}$$

In the second case, some back-tracking must occur

In the first case, we have

$$\bigcup_{n\in\mathbb{Z}} g^n(a) \underset{\text{isom.}}{\cong} \mathbb{R}$$

each $g^n(a)$ being an interval of $\mathbb{R}$ of length equal to length(a);



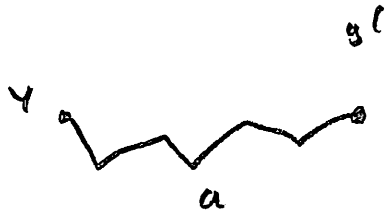
$g^2(a) \cap g(a) = g^2(y)$ since g is a homeomorphism.

and $g^2(a) \cap a = \emptyset$ since T is a tree.

Continue to see that $g^n(a) \cap g^{n'}(a) = \emptyset$ if $|n-n'|>1$

$$g^n(a) \cap g^{n+1}(a) = g^{n+1}(y)$$
$$g^n(a) \cap g^{n-1}(a) = g^n(y).$$

In the second case we have

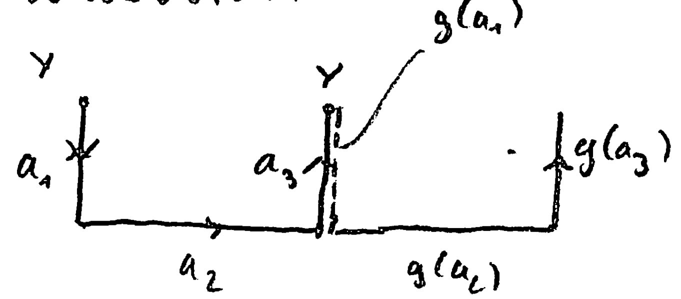


$g(a)$ is the unique path in T from $g(y)$ to $g^2(y)$

$\cdot\, g^2(y)$

Look at the unique path from y to $g^2(y)$. This is obtained from $a \cdot g(a)$ by reducing. Since a, $g(a)$ are reduced, the last part of a must be inverse to the first part of $g(a)$

so we obtain

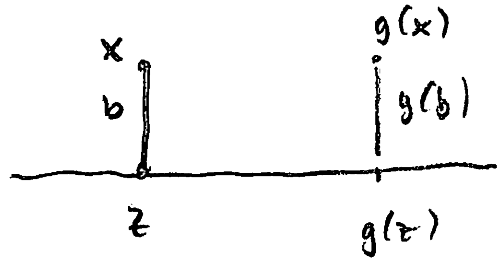


and we obtain an axis for g in $\bigcup_n g^n(a_2)$.

Clearly axes are axes.

If $A = \bigcup_n g^n(a)$ with a unique reduced path from y to $g(y)$ and $a \cap g(a) = g(y)$

and $x \notin A$. Let b be the unique shortest path from x to A. Then look at



to see that $d(x, g(x)) = d(z, g(z)) + 2d(x,z)$.

6.3.2 Remarks

(a) Define $\ell(g) = \min_{x \in T} \{ d(x, g(x)) \}$.

Then $\ell(hgh^{-1}) = \ell(g)$

(b) $T \longmapsto \{ \ell(c) : c \in \mathcal{C} \} \subset \prod_{\mathcal{C}} \mathbb{R}_{>0}$

$\mathcal{C}$ = set of conj-classes of elements of F_n

is an injection

Strong equivalence classes of free metric F_n-trees

into $\prod_C \mathbb{R}_{>0}$. (give it the product topology)

(c) This defines another topology on Out_n. The so called lengths topology. Again the identity map is a homeomorphism with

$(Out_n$, weak $)$ and $(Out_n$, Gromov-Hausdorff$)$

(a) is easy. The other statements not.

If A is an axis for g, then $h(A)$ is an axis for hgh^{-1}:

$$d(x, gx) = d(hx, hgx) = d(hx, (hgh^{-1})(hx))$$

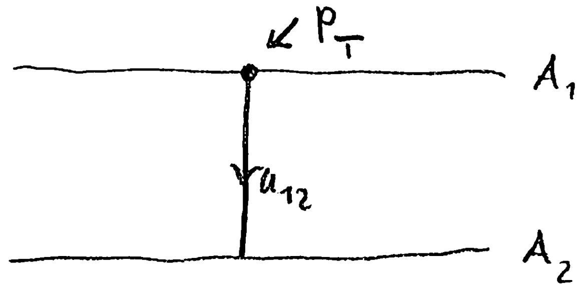
Thus $\ell_g \geq \ell_{hgh^{-1}} \quad (\geq \ell_{h^{-1}(hgh^{-1})h} = \ell_g)$

6.3.3 Choice of basepoint according to Guirardel-Levitt.

Recall: $n \geq 2$. Let $e_1, \dots, e_n$ be the standard basis of F_n. Let T be a free metric F_n-tree.

Let A_1, A_2 be the axes of e_1, e_2 in T

If $A_1 \cap A_2 = \emptyset$ consider the unique shortest path a_{12} from A_1 to A_2. Then we take $P_T = \iota(a_{12})$

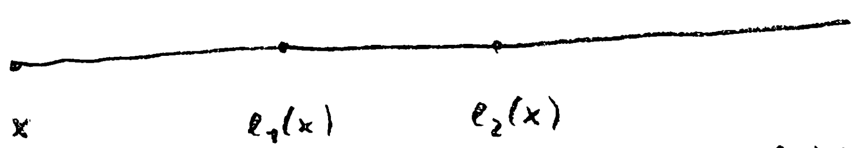


If $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cap A_2$ is a finite subintervall of A_1. Let A_1 be directed by saying that $e_1(x)$ comes after x for $x \in A_1$.

Then set P_T = final point of $(A_1 \cap A_2)$ on A_1.

We need to show that $A_1 \cap A_2$ is in fact finite.

If not, assume that it contains ~~the~~ a positive infinite ray of A_1 (if not exchange for our argument e_1 by e_1^{-1}), similarly it contains a positive infinite ray of A_2. So up to renumbering:



$e_2(x) \geq e_1(x)$

x a vertex. e_i is known once we have an axis and $\ell(e_i)$

so we may assume that $e_2(x) > e_1(x)$

If r_1 is the number of edges between x and $e_1(x)$

and r_2 is the number of edges between x and $e_2(x)$

then $r_2 - r_1$ is the number of edges between x and $e_1^{r_2}(x)$

and between x and $e_2^{r_1}(x)$

This implies $e_1^{r_2} = e_2^{r_1}$. Consequently, $e_1^{r_2}$ and $e_2^{r_1}$ commute. We have seen in an earlier exercise that then e_1 and e_2 commute, which is not the case.

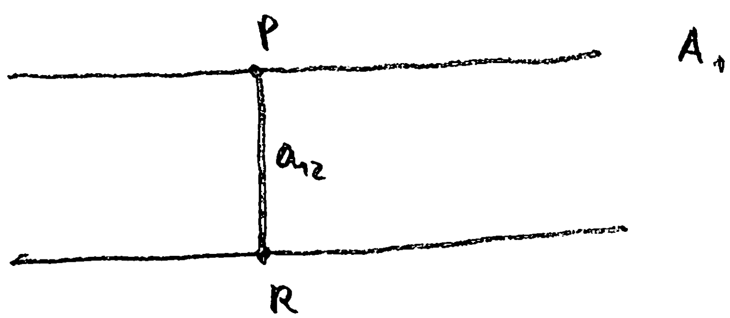
Thus: P_T depends on T only.

6.3.4 P_T depends continuously on T.

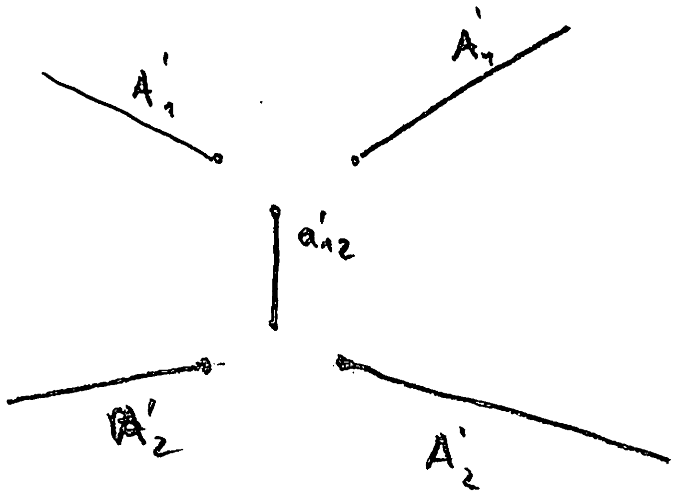
Proof. Look at $N_\varepsilon(T)$ as before.

1st Case $A_1 \cap A_2 = \emptyset$ in T

Then look at a_{12} in T

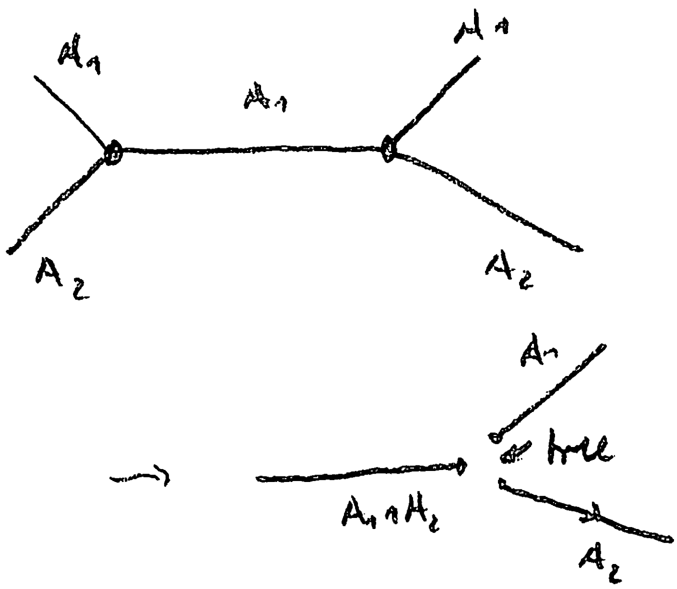


in T' P and R are replaced by small trees



and we see that $P_{T'}$ is some point in the tree "around" P_T, which is either mapped to P_T under $T' \to T$, or a point near P_T on the axis.

Second case



and the same argument applies.