Incomplete "Proof" of : „$\longleftarrow$" is continuous. We need to know that small neighborhoods in Gromov-Hausdorff topology map to small neighborhoods in the weak topology.

So let (T, φ) be a free metric F_n tree (well determined up to equivariant isometry). Let T/F_n have q edges $e_1, \dots, e_q$ of lengths $t_1, \dots, t_q$, $\sum_i t_i = 1$.

A neighborhood of (T, φ) in the weak topology always contains a neighborhood consisting of free metric F_n-trees (T', φ') s.t. T'/F_n contains a forest F s.t. each edge of F has length $< \delta$ and after collapsing F we obtain a graph which can be identified with T/F_n via a homeomorphism which is up to δ an isometry, i.e. if the homeomorphism maps the edge a_i to the edge e_i then $|\ell(a_i) - \ell(e_i)| < \delta$.

Thus shrinking the inverse image of F in T' equivariantly and adjusting the lengths of the remaining edges in T' equivariantly we get (T, φ) from (T', φ').

We now have to find $\{x_1, \dots, x_r\} \subset T$, $\{g_1, \dots, g_s\} \subset T$, $\varepsilon > 0$ such that $V_{x_1, \dots, x_r, g_1, \dots, g_s, \varepsilon}(T, \varphi)$ is contained in this neighborhood.

To find the appropriate points $x_0, \dots, x_r$ we first pass to $\Gamma_T = T/F_n$. Let T_0 be a maximal tree in Γ_T and $e_1, \dots, e_n$ the edges of Γ_T not in T_0. We think of these edges as being oriented (Then they correspond to a basis of $\pi_1(\Gamma_T, x_0)$, x_0 any vertex; but not necessarily to the standard basis of $\pi_1(\Gamma_T, x_0)$ provided by the action φ).

The inverse image of T_0 in T is a disjoint union of trees $g(\tilde{T}_0)$, $g \in F_n$, where $\tilde{T}_0$ is one component of the inverse image of our choice. Each $g(\tilde{T}_0)$ is projected isometrically onto T_0

To each e_i corresponds a unique lift, $\tilde{e}_i$ which is the lift with initial point $\iota(\tilde{e}_i)$ in $\tilde{T}_0$

Then there exists a unique $w_i \in F_n$ such that $w_i(\tau(\tilde{e}_i)) \in \tilde{T}_0$. (This point projects to $\tau(e_i)$ in T_0).

Let $x_1, x_2, \dots, x_k$ be the vertices of $\tilde{T}_0$ and $x_{k+1}, \dots, x_{k+n}$ the vertices $\tau(\tilde{e}_1), \dots, \tau(\tilde{e}_n)$.

Then we take for our definition of the neighborhood of (T, φ) the points $x_1, \dots, x_{k+n}$, the group elements $\overset{w_0 =)}{1}, w_1, \dots, w_n$ and $\varepsilon > 0$ such that $6\varepsilon < \mathrm{Min}\{\delta, \mathrm{Min}\{\ell(a) : a \in E_{\Gamma_T}\}\}$

We claim, that any (T', φ') in this neighborhood is contained in the δ-neighborhood of (T, φ) in the weak topology.

First notice that for every $x_1, \dots, x_k$ (the ~~points~~ vertices of $\tilde{T}_0$) we have at least three neighbors among $\{ w_i(x_j) : 0 \le i \le n,\ 1 \le j \le k+n \}$ in the tree T.

So let $x_{r_1}, x_{r_2}, x_{r_3}$ be neighbors of x_r. Then

$$d(x_{k_i}, x_{k_j}) = d(x_{k_i}, x_k) + d(x_k, x_{k_j}) \quad \text{for } i \neq j$$

Here some x_{r_i} could be of the form $w_a(x_b)$, $1 \le b \le k+n$.

Let $\tilde{x}_1, \dots, \tilde{x}_{k+n}$ be the points figuring in the Gromov-Hausdorff definition, and let $1 \le r \le k$

if $x_{r_1}, x_{r_2}, x_{r_3}$ are neighbors of x_r let $\tilde{x}_{r_1}, \tilde{x}_{r_2}, \tilde{x}_{r_3}$ be the corresponding points for $\tilde{x}_r$

(here $\tilde{x}_{r_i} = w_a(\tilde{x}_b)$, if $x_{r_i} = w_a(x_b)$).

If $d'(\tilde{x}_{r_i}, \tilde{x}_{r_j}) = d'(\tilde{x}_{r_i}, \tilde{x}_k) + d'(\tilde{x}_k, \tilde{x}_{r_j})$

then $\tilde{x}_r$ must be a vertex in T'.

Otherwise, without loss of generality we have $d'(\tilde{x}_{r_1}, \tilde{x}_{r_2}) < d'(\tilde{x}_{r_1}, \tilde{x}_r) + d'(\tilde{x}_r, \tilde{x}_{r_2})$

But

$$d(x_{r_1}, x_{r_2}) = d(x_{r_1}, x_r) + d(x_r, x_{r_2})$$

while all corresping $d'(\tilde{\ }, \tilde{\ })$ distances differ at most by ε from their counterparts in T.

Thus $$d'(\tilde{x}_{r_1}, \tilde{x}_{r_2}) > d'(\hat{x}_{r_1}, \tilde{x}_r) + d(\hat{x}_r, \tilde{x}_{r_2}) - 3\varepsilon$$

This means that on our shortes path from $\tilde{x}_{r_1}$ to $\tilde{x}_{r_2}$ we backtrack at most by a path of length $\frac{3}{2}\varepsilon$ on our path from $\tilde{x}_{r_1}$ to $\tilde{x}_r$ before moving off to $\tilde{x}_{r_2}$

Thus in a $\frac{3}{2}\varepsilon$-neighborhood of $\tilde{x}_r$ is a vertex of T'.

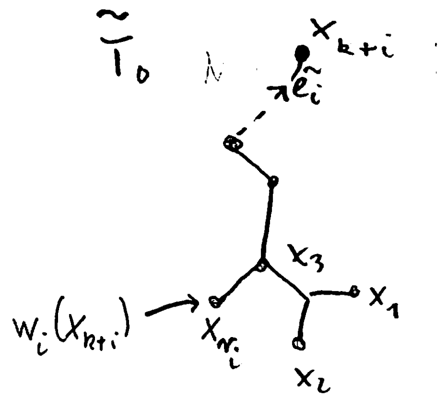
Consider for $r = 1, \dots, k$ the set of vertices V_r of T' of distance at most $\frac{3}{2}\varepsilon$ from $\tilde{x}_r$.

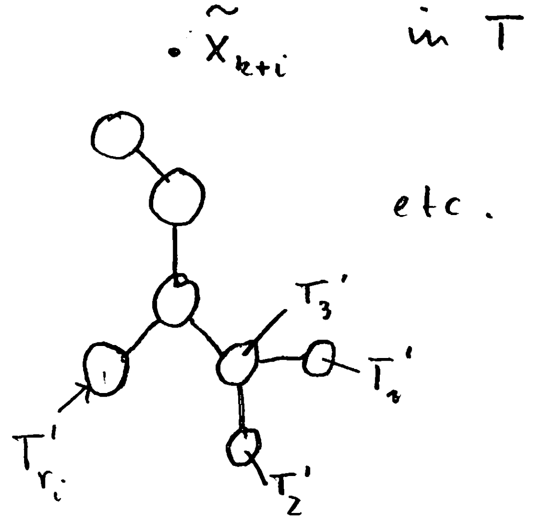
For $1 \le r, s \le k$, $r \neq s$ we have

$$d'(\tilde{x}_r, \tilde{x}_s) > d(x_r, x_s) - \varepsilon > 5\varepsilon$$

Thus, if $v_1 \in V_r$, $v_2 \in V_s$, $r \neq s$, $d'(v_1, v_2) > 3\varepsilon$,

Thus we get the following picture. For each $r = 1, \dots, k$, let T'_r be the convex hull of V_r in T', for each $1 \leq r, s \leq k$, $r \neq s$, such that there is an edge e_{rs} in $\tilde{T}_0$ from x_r to x_s there is a shortest path $\tilde{e}_{rs}$ in T' from T'_r to T'_s and if $(r,s) \neq (r',s')$ then $\tilde{e}_{rs}$ intersects $\tilde{e}_{r's'}$ if at all then only at one of the ends of $\tilde{e}_{rs}$. No $\tilde{e}_{rs}$ is a constant path. (Its length being $\geq d(x_r, x_s) - 5\varepsilon$)



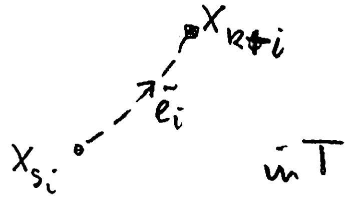


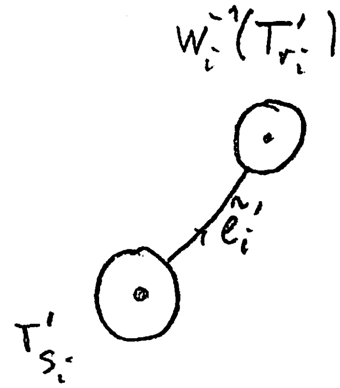
Now look at $w_i(\tilde{x}_{k+i})$. Then

$0 = d(x_{r_i}, w_i(x_{k+i}))$. Thus $d'(w_i(\tilde{x}_{k+i}), \tilde{x}_{r_i}) < \varepsilon$

i.e. $w_i(\tilde{x}_{k+i}) \in T'_{r_i}$. We transport T'_{r_i} via w_i^{-1} to a neighborhood $w_i^{-1}(T'_{r_i})$ of $\tilde{x}_{k+i}$. If $t(\tilde{e}_i) = x_{s_i}$ denote the shortest path from T'_{s_i} to $w_i^{-1}(T'_{r_i})$ by

$\tilde{e}_i'$.





$$x_{k+i} = w_i^{-1}(x_{r_i})$$

Passing to T'/F_n what we have constructed in T' up to here is a finite graph which is obtained (up to homeomorphism) from T/F_n by replacing some vertices of T/F_n by (small) trees. Consequently, denoting what we constructed,

$$\text{i.e.} \quad \bigcup_{r=1}^{k} T_r' \cup \bigcup_{r=k+1}^{k+i} w_i^{-1}(T_{r_i}')$$

$$\cup \bigcup_{1 \le r \ne s \le k} \tilde{e}_{rs} \cup \bigcup_{i=1}^{n} \tilde{e}_i'$$

by T'', then $\pi_1(T''/F_n) \cong F_n$ unless there is an element $g \in F_n$ with $g \notin \{1, w_1^{\pm 1}, \dots, w_n^{\pm 1}\}$ which maps a point of T'' to a point of T''.

[Here we simply assume that this is not the case. This follows basically from the fact that there is no such element mapping a point of $\tilde{T}_0 \cup \bigcup_{i=1}^{n} \tilde{e}_i$ to a point in this set]

Then there can be no further vertex in $\tilde{e}_i'$, $i=1,\dots,n$, apart from their endpoints. Otherwise, since vertices have valence ≥ 3, $\pi_1(T'/F_n)$ is isom. to a free group of rank $> n$.

6.3 Contracting Out_n. Intuitive Ideas.

If X is contractible then there exists by definition a homotopy $F: X\times I \longrightarrow X$ from id_X to a constant map, say c_{x_0}, $x_0 \in X$. Then for every $x \in X$ we have a path $t \overset{w_x}{\longmapsto} F(x,t)$ in X beginning at x, ending at x_0 and the paths depend continuously on x.

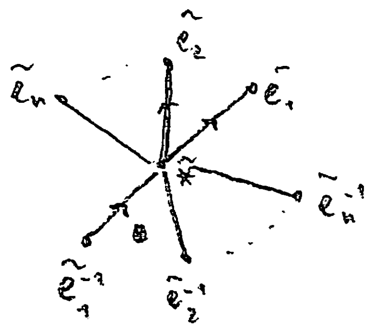
Usually, in concrete situations, there is a good candidate for x_0, and the paths w_x are to a certain degree natural.

Our natural candidate is the tree T_0, the universal cover of R_n with the standard action of $F_n = \pi_1(R_n, *)$ on T_0, and all edges of length $\frac{1}{n}$.

Now let (T, φ) be a representative of any element of Out_n.

The idea is to create a path T_t, $0 \leq t \leq \infty$ with $T_\infty = T$ from T_0 to T using a map $f: T_0 \to T$. The map is supposed to be equivariant and at least piecewise an isometry. For this we will actually have to change the lengths of edges in T_0. This will not cause much of a problem, because with this we vary T_0 only in an $(n-1)$-simplex Δ^{n-1} times $\mathbb{R}_{>0}$.

We first choose (for the moment any) point $P \in T$ and define $f(*) = P$. Since we want f to be equivariant f is now defined on all vertices of T_0 since the vertices of T_0 are $g(*)$, $g \in F_n$. Thus $f(g(*)) = g(P)$. We extend f to the edges of T_0 (corresponding to the standard generators and their translates)



by giving them the length

$$d'(P, e_i P)$$

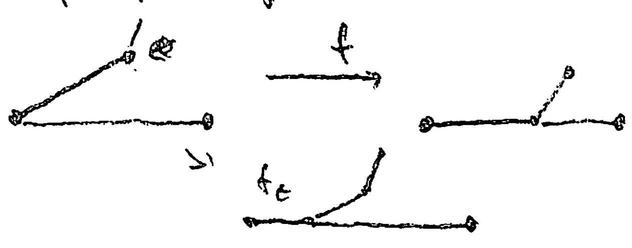
and mapping them isometrically.

Should f be an isometry, then we have $T \cong T_0$ and there is nothing to be done; we are somewhere in our simplex.

Otherwise, since f induces a homotopy equivalence

$$T_0/F_n \longrightarrow T/F_n$$

some folding must occur. The idea is to break up this folding into a continuous family f_t of foldings with $t \to \infty$ having $f_t \to f$.



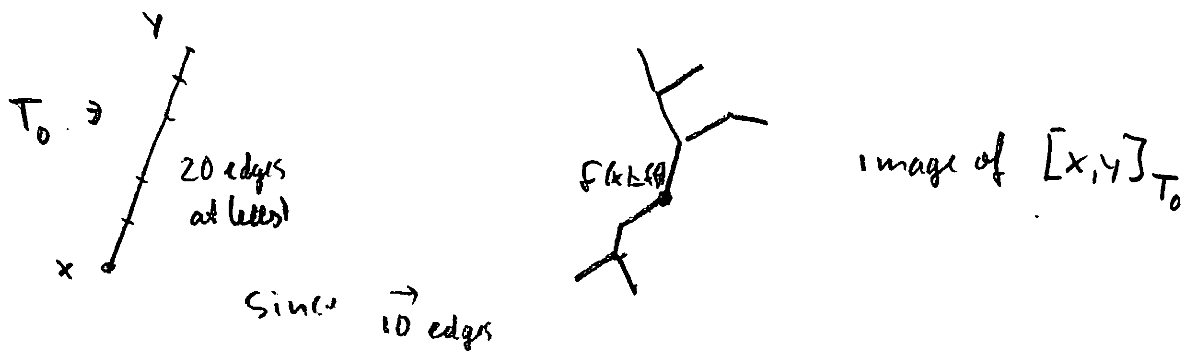
So T_t is some quotient of T_0 and T a quotient of T_t

and f_t the quotient map.

Now, how do we define T_t. We have to say, when two distinct points of T_0 are the same in T_t. A necessary condition is that they are the same in T.

So let $x \neq y$ in T_0 with $f(x) = f(y)$. Let

$$T_{xy} = f([x,y]_{T_0})$$



If $t \geq$ maximal distance of any point in $f([x,y]_{T_0})$ from $f(x)$ then x and y are identified in T_t.

give T_t the maximal metric on edges such that $T_0 \to T_t$ is a 1-Lipschitz map.

By looking at the quotients we know that (up to equivariance) only finitely many foldings occur. So there is a t_0 such that $T_t = T_{t_0}$ for all $t \geq t_0$