These pictures indicate one way to define a topology on Out_n:

Let Γ, Γ' be a metric graphs with $k+1$ edges and let $f: \Gamma \to R_n$ and $f': \Gamma' \to R_n$ be homotopy equivalences

We say that $[\Gamma, f]$ and $[\Gamma', f']$ belong to the same open k-simplex if there exists a homeomorphism $\varphi: \Gamma' \to \Gamma$ such that $f \circ \varphi \simeq f'$ (As always the vertices of Γ, Γ' have valence ≥ 3)

We identify the elements of Out_n in the same open k-simplex with the open k-simplex $\mathring{\Delta}^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i > 0, \sum t_i = 1\}$

as follows.

Pick some representative (Γ, f), label the edges by $e_0, e_1, \dots, e_k$, normalize all volumes of graphs to 1. Let (Γ', f') be any marked metric graph in the same k-simplex as (Γ, f), and let φ be a homeomorphism $\varphi: \Gamma' \to \Gamma$ with $f' = f \circ \varphi$

Then map $[\Gamma', f'] \in Out_n$ to $\left(\ell(\varphi^{-1}(e_0)), \dots, \ell(\varphi^{-1}(e_k)) \right) \in \mathring{\Delta}^k$.

This is well defined, since any homeomorphism of Γ homotopic to id_Γ fixes all vertices and maps edges to themselves by an orientation preserving homeomorphism (Compare 6.1.10 and Problems 1 & 2 on Sheet 13)

In particular, (Γ, f) gets mapped to

$$(\ell(e_0), \ldots, \ell(e_k))$$

Also notice, if $[(\Gamma, f)] = [(\Gamma', f')]$ in Out_n then there exists an isometry $\varphi : \Gamma' \to \Gamma$ such that $f' = f \circ \varphi$ and then

$$\ell(\varphi^{-1}(e_i)) = \ell(e_i)$$

When is an open $(k-1)$-simplex of Out_n in the boundary of the open k-simplex containing $[(\Gamma, f)]$?

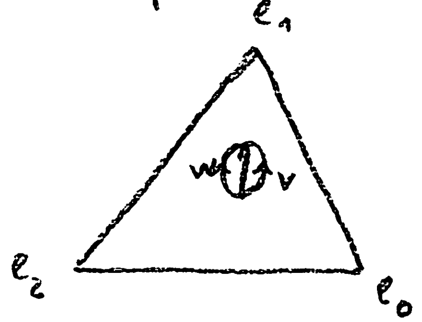
Let $0 \le i \le k$, and let in Γ the length of the edge e_i go to 0, while keeping the lengths of the other edges positive and converging to positive values. We then obtain a new metric graph $\bar{\Gamma}$. If $\pi_1(\bar{\Gamma}) \cong F_n$, then for some homotopy equivalence $\bar{f} : \bar{\Gamma} \to R_n$ $[\bar{\Gamma}, \bar{f}]$ is in an open $(k-1)$-simplex. The best way to determine for which homotopy class $[\bar{f}]$ we have $[\bar{\Gamma}, \bar{f}]$ in the i-th boundary simplex of our open k-simplex is as follows:

Let $T \subset \Gamma$ be a maximal tree containing e_i (notice that e_i cannot be a loop, if $\pi_1(\bar{\Gamma}) \cong \pi_1(\Gamma)$) and pick a representative f of $[f]$ which maps T to the basepoint of R_n. The edges a_j outside T are mapped to a basis of $\pi_1(R_n)$. Shrinking e_i gives a new tree $\bar{T}$, maximal in $\bar{\Gamma}$. Now pick $\bar{f}$ to be constant on $\bar{T}$ $(= v_0)$, and map the edges not in $\bar{T}$ as before.

Thus Out_n becomes a union of open simplices in a simplicial complex $\Delta(\mathrm{Out}_n)$ and Out_n is given the subspace topology ~~of~~ $\Delta(\mathrm{Out}_n)$.

The corresponding topology of Out_n is called the <u>weak topology</u>. The name comes from the fact that a set A in a simplicial complex K is closed iff $A \cap \sigma$ is closed in σ for each closed simplex σ of K.

Going back to Out_2: Let us agree to draw open 2-simplices by

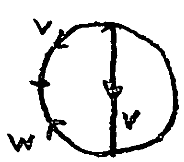


here the non-labeled edge gets mapped by f to the basepoint and the 2 labeled edges are mapped to the basis $\{w, v\}$ of $\pi_1(R_n, *)$.

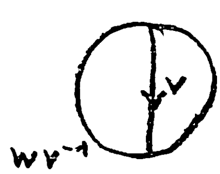
The edge on the left, center, right correspond to e_0, e_1, e_2

Thus if above the middle edge is shrunk to a point we end up on $e_2\,e_0$ with

If we want to shrink the right edge, notice that



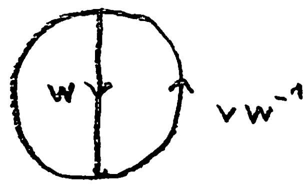
i.e.



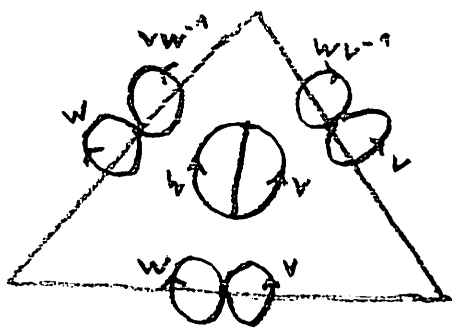
defines a map homotopic to f since it maps  to $wv^{-1}v = w$ and  to v

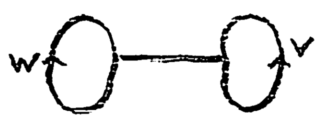
Then we can safely collapse the right edge.

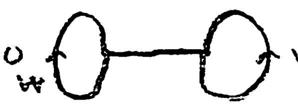
Similarly, for the left edge: take



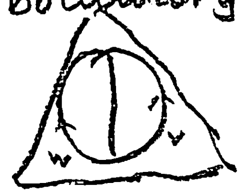
and then collapse. We indicate all this often by

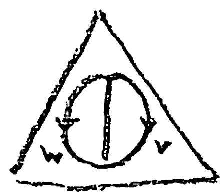


For 

the central edge is always the maximal tree, and you should have no problem collapsing it. So 

has only one open 1-simplex in its boundary.

Also note that in any interior point of 

we can always replace it by 

simply by reflecting on the horizontal axis, an isometry everywhere. We get thus the <u>same</u> point in Out n.

On more symmetric points like perpendiculars to edges through the opposite vertex or in the barycenter we ~~have~~ more isometries

One last remark before continuing:

If $e_1, \dots, e_t$ are seperating edges of Γ, we can safely collaps them (they appear in any maximal tree)

So we obtain a homotopy equivalent Out_n if we only consider metric Γ's without seperating edges. The corresponding space is sometimes called

<u>Reduced Out_n</u>.

Notice also that $Out(F_n)$ acts on Out_n by mapping open k-simplices homeomorphically to open k-simplices for all k. Also notice that

$n-1 \leq k \leq 3n-4$, since any Γ htpy equiv. to R_n contains at least n edges, and we have seen that Γ has at most $3n-3$ edges if the valence of all vertices is at least 3.

Back to the second definition of Out_n via actions of F_n on trees.

Thus T is a (metric) tree; valence of vertices ≥ 3

F_n acts freely and isometrically with finite quotient graph. $\Gamma_T := T/F_n$. Then $\pi_1(\Gamma_T, *)$ is canonically isomorphic to F_n once we fix $\tilde{*} \in T$ and $*$ the image of $\tilde{*}$ in Γ_T:

If $g \in F_n$ let $\tilde{w}_g$ be the unique reduced path in T from $\tilde{*}$ to $g(\tilde{*})$. Then image w_g in Γ_T is a reduced loop starting at $*$. $g \mapsto [w_g]$ is the desired isomorphism.

6.1.15 <u>(Repeat of definition)</u> (a) A free metric F_n-tree is a metric tree T together with a free isometric F_n-action s.t. $T/_{F_n}$ is a finite graph. Furthermore the valence of every vertex of T (or of $T/_{F_n}$) is at least 3.

(b) Two free metric F_n-trees (T, φ), (T', φ') are considered to be equal (i.e. are equivalent) if there exists an equivariant isometry $\psi: T \to T'$ i.e. an isometry s.t. $\forall g \in F_n$

$\forall x \in T: \psi(\varphi(g,x)) = \varphi'(g, \psi(x))$, where $\varphi: F_n \times T \to T$ and $\varphi': F_n \times T' \to T'$ are the respective actions.

i.e. $\forall g \in F_n$ the diagram

$$\begin{array}{ccc} T & \xrightarrow{\psi} & T' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ T & \xrightarrow{\psi} & T' \end{array}$$

commutes where g describes the isometries on T resp. T' given by the resp. actions.

(c) Points of Out_n are equiv. classes of free metric F_n-trees.

6.1.16 Action of Out F_n on Out_n. If $[\alpha] \in \mathrm{Out}(F_n)$ with representative $\alpha \in \mathrm{Aut}\, F_n$ define

$$[T, \varphi] \cdot [\alpha] := [T, \varphi \circ (\alpha \times \mathrm{id})]$$

Remark: Is well defined: here it is useful to introduce a little notation.

Define for $g \in F_n$ the map $x \longmapsto \varphi(g, x)$ simply by $x \longmapsto gx$. Let $\alpha = c_g$, the inner autom. $h \longmapsto ghg^{-1}$. Then we claim that $g : T \longrightarrow T$ is an isometry (clear) which is equivariant with respect to the actions φ on domain and $\varphi \circ (c_g \times \mathrm{id}_T)$ on range.

i.e. $\forall h \in F_n$

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "h"'] & T \arrow[d, "c_g(h)"] \\ T \arrow[r, "g"] & T \end{tikzcd}$$

commutes, which is obvious

Recall, that there is an isomorphism between Out F_n and the group of homotopy classes of homotopy equivalences of R_n
(See Point 4 on Background Info Sheet II)

We get from $[(\Gamma, f)]$ of first definition to $[T_\Gamma, \varphi_f]$ in 2^{nd} definition by T_Γ = univ. cover of Γ, and φ_f to be the action given by $\sigma \circ (f_* \times id_{T_\Gamma})$, where

$\sigma : \pi_1(\Gamma, x_0) \times T_\Gamma \longrightarrow T_\Gamma$ is the standard action of $\pi_1(\Gamma, x_0)$ on T_Γ and $f_* : F_n = \pi_1(R_n, *) \to (\Gamma, x_0)$ is the isom. induced by f_*.

You should have no problem defining the inverse map.

6.2 Three topologies on Out_n.

6.2.1 1^{st}. Weak topology as defined above (and then transferred to the second definition.

6.2.2 2^{nd} Gromov-Hausdorff definition.

This is an adaption of the Gromov-Hausdorff distance between metric spaces to our particular situation.

Let $\mathcal{T}_n$ be the set of free metric F_n-trees.

Let $(T, \varphi) \in \mathcal{T}_n$ and let $\{x_1, \dots, x_r\} \subset T$, $\{g_1, \dots, g_s\} \subset F$ be finite (ordered) sets. For $\varepsilon > 0$ let

$$V_{x_1, \dots, x_r, g_1, \dots, g_s, \varepsilon}(T, \varphi) := \Big\{ (T', \varphi') :$$

$$\exists\ \tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_r \in T' \text{ s.t.}$$

$$| d_{T'}(\tilde{x}_i, g_j \tilde{x}_k) - d_T(x_i, g_j x_k) | < \varepsilon$$

for all i, j, k. $\Big\}$

Take these sets as a neighborhood basis for (T, φ).

6.2.3 <u>Remark:</u> If $\psi : T \longrightarrow T'$ is an equivariant isometry then T' is in any neighborhood of T. Actually $\forall x_1, \dots, x_r \in T$, $g_1, \dots, g_s \in F_n$ we have

$$d_{T'}\left(\psi(x_i), g_j(\psi(x_k))\right) = d_T(x_i, g_j x_k)$$

So this topology induces a topology on Out_n in the second definition.

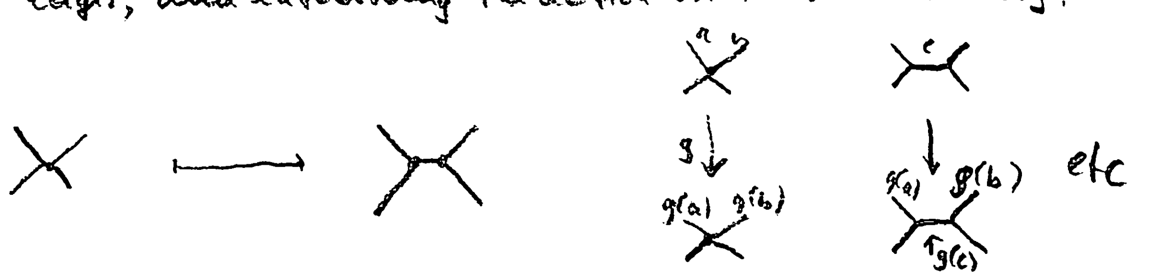
We call this the Gromov-Hausdorff topology.

<u>6.2.4</u> <u>Proposition:</u> The identity is a homeom.

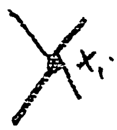
$$(\mathrm{Out}_n; \text{weak}) \xrightarrow{\ id\ } (\mathrm{Out}_n; G\text{-}H).$$

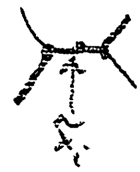
"<u>Proof</u> of : $\longrightarrow$ " is continuous.

The weak topology is given by length functions on edges of T/F_n. In the tree this means changing lengths of edges (equivariantly) and possibly introducing short edges, and extending the action in the obvious way.



In the first case pick $\tilde{x}_i$ to be the old x_i

If $x_i \longrightarrow$ 



Given $x_1, \ldots, x_r$, $g_1, \ldots, g_s$, ε it is clear that we can make the change in length parameters small enough to land in $V_{x_1, \ldots, x_r, g_1, \ldots, g_s, \varepsilon}(T)$.