6.1 Outer Space Out_n as a set: two definitions

Since $\mathrm{Aut}(F_1) = \mathrm{Out}(F_1) = \mathbb{Z}/2\mathbb{Z}$ is not really mysterious, we only deal with $n \geq 2$.

The set Out_n will be defined as equivalence classes of certain objects in both definitions.

<u>6.1.1 Definition</u>. A <u>metric graph</u> (Γ, d) is a graph where each geometric edge, i.e. each edge pair $\{e, \bar{e}\}$ is given a positive length $\ell(e)$ $(= \ell(\bar{e}))$

6.1.2 <u>Remark</u>: This will define a length for any path in the sense of graphs, but also a length for any path in the sense of (reasonable) continuous path $\omega : [0,1] \longrightarrow G(\Gamma)$, where $G(\Gamma)$ is the geom. realization of Γ. As before, we do not (really) distinguish between Γ and $G(\Gamma)$. We consider a path to be reasonable if the restriction of ω to the components of $\omega^{-1}(e)$, is rectifyable when considered as a map into $(0, \ell(e))$. The paths which we consider will be piecewise affine in the obvious sense, i.e.

if $[a,b] \subset [0,1]$ and $\omega[a,b] \subset e$, then $[a,b]$ is the union of finitely many intervals J_k such that $\omega|_{J_k} : J_k \longrightarrow e \, (= [0, \ell(e)])$ is affine.

6.1.3 Remark: Then any two points x, y in a connected metric graph have a distance given by the length of a shortest path from x to y in Γ.

6.1.4 Remark. If we define a *geodesic* in (Γ, d) to be a path $\omega: [0, a]$ s.t. for all $t \in [0,a]$

$t = d_1(\omega(0), \omega(t))$, then any two points in Γ can be connected by a geodesic. One calls (Γ, d) a geodesic metric space.

Recall the graph R_n consisting of 1 vertex x_0 and n edge pairs (the rose with n petals). We identify $\pi_1(R_n, x_0)$ with F_n. Recall that this amounts to choosing an orientation v of R_n and labeling the edges by $x_1, \dots, x_n$.

$(F_n := F(x_1, \dots, x_n))$.

6.1.5 Definition: (a) A *marked metric graph* (Γ, f) consists of a metric graph Γ together with a homotopy equivalence (of top-spaces)

$$f : R_n \longrightarrow \Gamma$$

where Γ is a metric graph with $\mathrm{val}(v) \geq 3$ for every vertex v.

(b) Two marked metric graphs (Γ, f) and (Γ', f') are equivalent, if there exists a homothesy $\varphi : \Gamma \to \Gamma'$ such that $\varphi \circ f$ and f' are homotopic. (A homothesy is a map $\varphi : (X, d) \to (X', d')$ s.t. $d'(\varphi(x), \varphi(y)) = c \cdot d(x, y)$ for a fixed $c > 0$)

(c) The points of Out_n (first definition) are equivalence classes of marked metric graphs

Some remarks are in order

6.1.6 (a) Up to homotopy any homotopy equivalence is determined by a homotopy inverse. i.e. If $f : X \to Y$ is a homotopy equivalence and $g : Y \to X$ a homotopy inverse. Let f' be a homotopy inverse of g. Then $f' = f' \circ id_X \simeq f' \circ (g \circ f)$

$$= (f' \circ g) \circ f \simeq id_Y \circ f = f \text{ , i.e. } f' \simeq f.$$

(b) A marking gives us an isomorphism $f_* : F_n \to \pi_1(\Gamma)$ (up to inner automorphism, since we have no base point) Since $F_n = \pi_1(R_n, x_0)$.

(c) Since we may replace within the equivalence class (Γ, f) with (Γ, f') where $f' \simeq f$ we may assume that f maps the edge x_i to a reduced loop in the graph sense. Call this loop $f(x_i)$. Then Giving each x_i in R_n length $\frac{1}{n}$, we can subdivide x_i into edges mapped by

$f(x_i)$ via a homothesy to edges of Γ.

6.1.7 Definition. $\operatorname{Aut} F_n$ acts on points of Out_n by precomposition (action on the right). I.e. let $\varphi \in \operatorname{Aut} F_n$. Then there is a unique homotopy class f_φ of base point preserving map $(R_n, x_0) \to (R_n, x_0)$ with $f_{\varphi}: \pi_1(R_n, x_0) \to \pi_1(R_n, x_0)$ equal to φ.

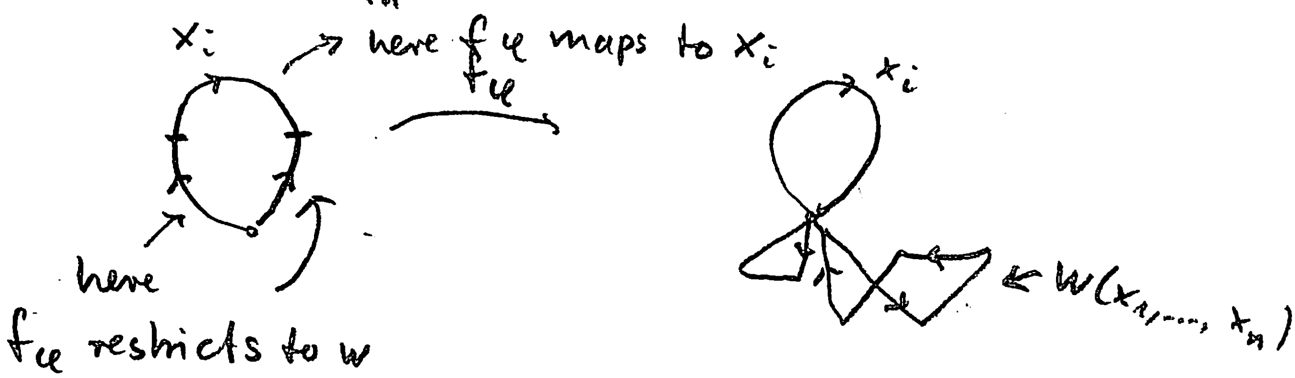
Then let $(\Gamma, f) * \varphi := (\Gamma, f \circ f_\varphi)$.

6.1.8 Remark. Let $\varphi = c_x$, $x = w(x_1, \dots, x_n) \in F_n$,

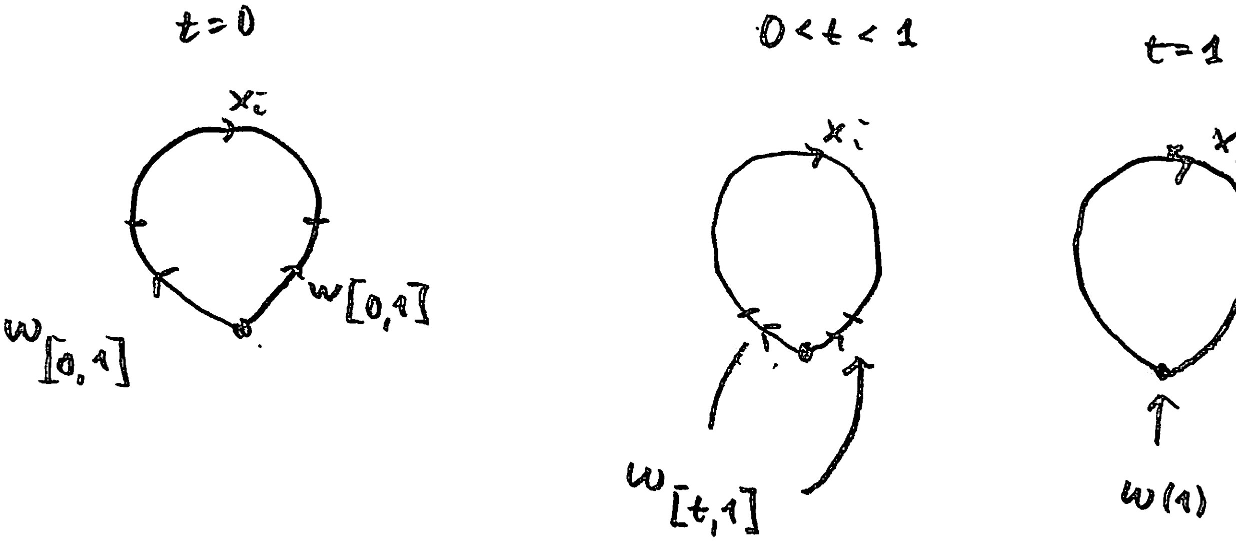
w a reduced word.

Then $f_\varphi(x_i) = w(x_1, \dots, x_n)\, x_i\, w(x_1, \dots, x_n)^{-1}$

As a map $f_\varphi : R_n \to R_n$ is homotopic to id_{R_n} (basepoints are allowed to be moved)



The homotopy pulls the basepoint along $w(x_1, \dots, x_n)$ until we reach the identity:



Thus, in particular

6.1.9 The action of $\mathrm{Aut}(F_n)$ on Out_n factors through an action of $\mathrm{Out}(F_n)$. That is the one we consider

6.1.10 What is the isotropy group of (Γ, f) (we write (Γ, f) although we mean its equivalence class) ?

<u>Claim</u>: $\mathrm{Out}(F_n)_{(\Gamma, f)} \cong (\mathrm{Isom}(\Gamma)) \subset \underset{\substack{\uparrow \\ \text{finite}}}{\mathrm{Aut}(\Gamma)}$

<u>Proof.</u>

Let $\varphi : R_n \to R_n$ be a homotopy equivalence (Homotopy classes of these correspond to elements of $\mathrm{Out}(F_n)$) Then

$(\Gamma, f) \cdot \varphi \simeq (\Gamma, f)$ iff $\exists$ isometry φ' : $\Gamma \to \Gamma$ s.t. $\varphi' \circ f \simeq f \circ \varphi$

(Here we normalize each graph to have volume $(= \sum_{\{e, \bar{e}\}} \ell(e)) = 1$.)

i.e. $\varphi' \simeq f \circ \varphi \circ g$, where g is a htpy inverse of f

If φ'' is another such isometry then $\varphi'' \simeq \varphi'$,

i.e. $\varphi''^{-1} \circ \varphi' \simeq id_{\Gamma}$. Let ψ be an isometry of Γ homotopic to id_{Γ}.

Subclaim (Exercise): $\psi \equiv id$.

Hint: Let Γ be a finite graph with $val(v) \geq 3$ for all vertices v. Let $\psi : \Gamma \to \Gamma$ be a homeomorphism which is homotopic to id_{Γ}. Then ψ fixes every vertex and maps edges homeomorphically and orientation preserving onto themselves (use the fact that ψ induces the identity on the first homology group of Γ)

To pass to the second definition of the set Out_n notice that the universal cover of a connected metric graph Γ is a metric tree $\tilde{\Gamma}$ on which $\pi_1(\Gamma, x_0)$ acts freely by isometries. The homotopy equivalence $f: R_n \to \Gamma$ identifies $F_n = \pi_1(R_n)$ up to an inner automorphism of F_n with $\pi_1(\Gamma, x_0)$. Thus we have a metric tree T together with a free isometric action of F_n such that the orbit graph is a finite metric graph.

As before, we assume that the valence of each vertex of T (= valence of some vertex in Γ) is at least 3 for every vertex of T.

Again we make the

<u>6.1.11 Definition</u>. (a) A free metric F_n-tree is a metric tree T together with a free isometric action of F_n such that the orbit graph T/F_n is finite with all vertices having valence at least 3.

(b) Two free metric F_n-trees are (strongly) equivalent if there exists an equivariant isometry from one to the other.

They are called equivalent if there exists an equivariant homothesy between them.

(c) Points of Out_n are equivalence classes of free metric F_n-trees. The set of strong equivalence classes will be denoted by $\mathcal{D}_n$ (the deformation space associated to $Out(F_n)$). Notice $Out_n = \mathcal{D}_n / \mathbb{R}_{>0}$

(d) $Aut\, F_n$ acts on $\mathcal{D}_n$ by precomposition, i.e.

If $m : T \times F_n \longrightarrow T$ is the given free metric action of F_n on T, and $\varphi \in Aut\, F_n$. Then $m \cdot \varphi := m \circ (id_T \times \varphi)$.

6.1.12 Notice: If $\varphi = c_{x^{-1}}$ is an inner autom. of F_n then

$$m_x : T \longrightarrow T$$
$$t \longmapsto m(t,x)$$

is an isometry such that

$$m_x(m_y(t)) = m_{yx}(t) = m_{x^{-1}yx}\, m_x(t)$$

so that m_x is equivariant with respect to the actions m and $m \cdot \varphi$. $\square$

You should have no problem to show

<u>6.1.13 Remark</u> The two definitions of Out_n agree, i.e. there is an obvious bijection between the sets Out_n defined in the two ways.

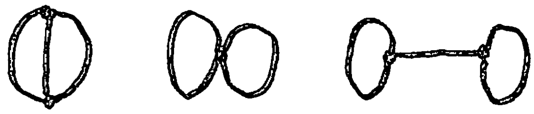
<u>6.1.14 Example:</u> We take a look at Out_2.

(a) <u>Notice</u>: For any connected Γ with $\pi_1(\Gamma) \cong F_n$ we have $\# V_\Gamma - \#\{\text{edge pairs}\} = 1-n$. Thus if every vertex has valence ≥ 3 we have

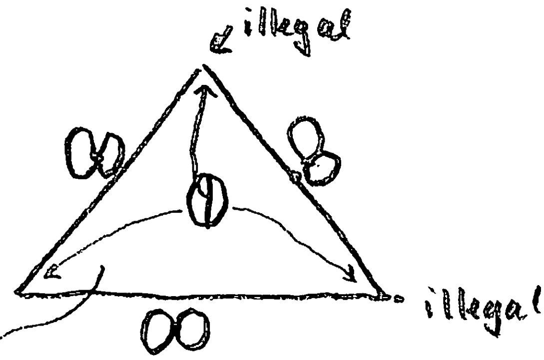
$$\#\{\text{edge pairs}\} = \frac{1}{2} \sum_{v \in V_\Gamma} \text{val}(v) \geq \frac{3}{2} \# V_\Gamma \quad \text{so that}$$

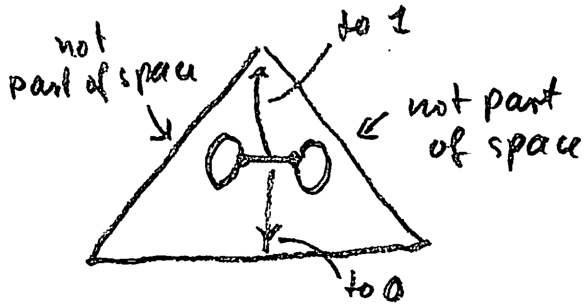
$$1-n \leq -\frac{1}{3}\#\{\text{edge pairs}\}, \text{ i.e. } \#\{\text{edge pairs}\} \leq 3n-3$$

(b) Thus for $n = 2$ Γ can be only one of the following 3 up to isomorphism.



normalizing the volume to 1 we see that up to isometry the first and last graphs correspond to 2-simplices, the second to a one simplex





length of corresp edge goes to 1

where the barycentric coordinates corresponds to the lengths of

Now each edge and triangle should come equipped with a homotopy equivalence from R_2. This time it is better to describe one to R_n by indicating a maximal tree