6. Outer Space

Before we introduce the space that will occupy our attention for the rest of this semester:

Definition of terms:

6.1 Def. G group, $g \in G$. Then $x \longmapsto g x g^{-1}$, $x \in G$, is an automorphism of G. We denote it by c_g (conjugation of elements of G by g). The automorphisms c_g are called inner automorphisms.

<u>6.2 Remark</u>. (a) The inner autom. of G form a subgroup of $\operatorname{Aut} G$ (= group of automorphisms of G) denoted by $\operatorname{Inn} G$

(b) $\operatorname{Inn} G \subset \operatorname{Aut} G$ is normal: If $\alpha \in \operatorname{Aut}(G)$ then

$$\alpha c_g \alpha^{-1} = c_{\alpha(g)}$$

(c) $G \longrightarrow \operatorname{Aut}(G)$, $g \longmapsto c_g$, is a homomorphism

Its kernel is the center C_G of G; $C_G = \{g \in G \mid gx = xg \text{ for all } x \in G\}$

<u>6.3 Def.</u> The quotient group $\operatorname{Aut}(G)/\operatorname{Inn}(G)$ is called the group of <u>outer automorphisms of G</u> and is denoted by $\operatorname{Out}(G)$. An element of $\operatorname{Out}(G)$ we will sometimes call an outomorphism. (Non-standard notation)

<u>6.4 Remark</u>. Passing from Aut to Out is similar to forgetting a basepoint for the fundamental group: if X is path-connected, then for any $x_0, x_1 \in X$ the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are canonically isomorphic up to inner automorphism of $\pi_1(X, x_0)$ (or

of $\pi_1(X,x_1)$.)

To see this, choose a path w from x_1 to x_0 in X.

$$\text{Then}\quad \begin{array}{ccc} [a] & \xmapsto{h_{[w]}} & [w*a*w^-] \\ \uparrow & & \uparrow \\ \pi_1(X,x_0) & & \pi_1(X,x_1) \end{array} \quad \text{is an isomorphism}$$

with $[b] \longmapsto [w^-*b*w]$ as its inverse.

Choosing a second path u from x_1 to x_0 we get a second isomorphism $h_{[u]} : \pi_1(X,x_0) \longrightarrow \pi_1(X,x_1)$

$$\text{and}\quad h_{[u]}^{-1} \circ h_{[w]} ([a]) = [u^- * w * a * w^- * u]$$

$$= c_{[u^-*w]}([a]), \quad \text{with } [u^-*w] \in \pi_1(X,x_0)$$

$$\text{similarly}\quad h_{[u]} \circ h_{[w]}^{-1}([b]) = c_{[u*w^-]}([b]), \quad \text{with}$$

$$[u*w^-] \in \pi_1(X,x_1).$$

<u>6.5 Remark.</u> In topology one is interested in Homeo(X) the group of homeomorphisms of X. There is a good topology on Homeo(X), in particular if X is locally compact:

if X, Y are top. spaces, and Map(X,Y) the set of continuous maps from X to Y, a subbasis of the so called compact-open topology is given by the sets $V(K,O)$, $K \subset X$ compact, $O \subset Y$ open, with

$V(K,O) = \{ f \in \text{Map}(X,Y) : f(K) \subset O \}$

For example, the set of homotopy classes of (contin.) maps from X to Y is the set of path components of $Map(X,Y)$ with this topology.

The set of path-components of $Homeo(X)$ is the set of <u>isotopy</u> classes of homeomorphisms of X.

Here $f, g \in Homeo(X)$ are isotopic if there is a level preserving homeo $F: X \times I \longrightarrow X \times I$ with $F(x,0) = (f(x), 0)$, $F(x,1) = (g(x), 1)$

(Level preserving means $F(X \times \{t\}) \subset X \times \{t\}$ $\forall t \in [0,1]$).

<u>6.6 Remark.</u> For all path-connected spaces X, each map f homotopic to id_X induces an inner automorphism of $\pi_1(X)$, when we take 6.4 into consideration:

Pick some base point x_0, let f_t, $0 \le t \le 1$, be ~~the~~ some homotopy from id_X to $f = f_1$.

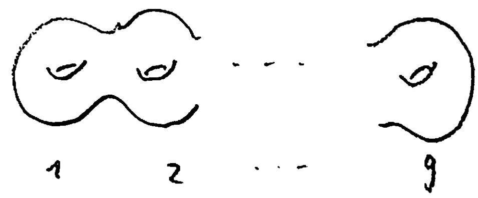
Let $w(t) := f_t(x_0)$. Then

$$[a] \longmapsto [w * f(a) * w^-]$$ is the identity

map of $\pi_1(X, x_0)$ for any path u from $f(x_0)$ to x_0 the map $[a] \to [u^- * f(a) * u]$ is an inner automorphism of $\pi_1(X, x_0)$.

Actually, there are even stronger reasons to study $Out(G)$ instead of $Aut(G)$.

6.5 <u>Remark</u>: (a) If Σ_g is the closed orientable surface of genus g.



Then $$Homeo(\Sigma_g)/Iso(\Sigma_g) \cong Out(\pi_1 \Sigma_g)$$

where $Iso(\Sigma_g)$ are the homeom. of Σ_g isotopic to id_{Σ_g}

(b) This is also true in the differentiable category.

To sum up: we should be interested in $Out(F_n)$ and not only $Aut(F_n)$

To study groups G it often helps to have spaces X with "nice" properties on which G acts in a "nice" way.

One example: invariants of G show up as homotopy invariants of certain spaces on which G acts.

Example: Homology of groups

Let X be a contractible space on which G acts freely ($g \in G$ and there is $x \in X$ with $g(x) = x$, then $g = 1_G$) and discontinuously (for any $x \in X$ there is a nbhd U of x s.t. for any $g \in G$ $g(U) \cap U = \emptyset$).

Then the quotient map $X \xrightarrow{p} X/G$ is a covering. Thus X is the universal cover of X/G and $\pi_1(X/G) \cong G$.

If X is 'nice enough', for example if X/G is a CW-complex, then for any other such space Y we have that X/G and Y/G are homotopy equivalent

More holds: fixing some $\tilde{x}_0$ in X ($\tilde{y}_0$ in Y) defines an isomorphism $\pi_1(X/G, p(x_0)) \longrightarrow G$; similarly

$$\pi_1(Y/G, p(y_0)) \longrightarrow G$$

So we may identify these groups.

Now: For any homom. $\varphi : G \longrightarrow G$ we get a map $f(\varphi) : X/G \longrightarrow Y/G$ <u>unique up to homotopy preserving $p(x_0)$, $p(y_0)$</u> inducing φ, or on the universal cover a map

$$\tilde{f}_\varphi : X \longrightarrow Y \quad \text{s.t.} \quad \tilde{f}_\varphi(g \cdot x) = \varphi(g)\tilde{f}_\varphi.$$

In particular the homology and cohomology groups of X/G depend only on G and not the choice of X and free, discontinuous action of G.

6.6 (Def.) These groups are called the homology and cohomology groups of G.

6.7 Remark There are algebraic definitions of these (Co)-homology groups. Having a good space X at hand might help with calculations.

6.8 Simple example $\mathbb{Z}^n$ acts via integral translations freely and discontinuously on $\mathbb{R}^n$

$(i_1, \dots, i_n) \cdot ((x_1, \dots, x_n)) = (x_1 + i_1, \dots, x_n + i_n)$,

$(i_1, \dots, i_n) \in \mathbb{Z}^n$, $(x_1, \dots, x_n) \in \mathbb{R}^n$.

$\mathbb{R}^n / \mathbb{Z}^n \cong T^n \ (= (S^1)^n)$ is the n-dimensional torus

It is easy to calculate $H_i(T^n)$ (i.e. $H_i(\mathbb{Z}^n)$) using the Künneth-theorem which describes the homology of a product space $X \times Y$ via the homology of X and Y.

Since $\dim T^n = n$ we have $H_i(\mathbb{Z}^n) = 0$ for $i > n$.

6.9 Cyclic groups: $G = \mathbb{Z}/k\mathbb{Z}$

$X := S^\infty$ $\quad S^\infty = \bigcup_{i=1}^{\infty} S^{2i-1}$, S^{2i-1} the unit sphere in $\mathbb{C}^i$, so S^∞ the unit sphere in

$$\mathbb{C}^\infty = \{(z_1, z_2, z_3, \dots) \mid z_i \in \mathbb{C} \,;\, \exists r \text{ s.t. } z_i = 0 \ \forall i > r\}$$

Thus $S^\infty = \{(z_1, \dots,) \in \mathbb{C}^\infty \mid z_1\bar{z}_1 + \cdots = 1\}$

this is a finite sum.

$$= \{(x_1, x_2, \dots) \in \mathbb{R}^\infty \mid x_1^2 + x_2^2 + \cdots = 1\}$$

It is easy to see that S^∞ is contractible using the fact that $S^n \hookrightarrow S^{n+1}$ can be contracted keeping the basepoint $(1, 0, \dots)$ fixed by pulling it over the positiv hemisphere

$$((x_1, \dots, x_{n+1}, 0, \dots), t) \to \Big((1-t)x_1 + t, (1-t)x_2, \dots, (1-t)x_{n+1},$$
$$\sqrt{1 - \Big(\big((1-t)x_1 + t\big)^2 + (1-t)^2 \sum_{i=2}^{n+1} x_i^2\Big)},$$
$$0, 0, \dots\Big)$$

keeping the basepoint $(1, 0, 0, \dots)$ fixed and patching these contractions coherently together.

G is isomorphic to the multiplicative group of k-th roots of 1, a subgroup of $S^1 \subset \mathbb{C} \setminus \{0\}$.
Let ξ be a primitive k-th root of 1, i.e. a generator of G. We let ξ act on $\mathbb{C}^\infty$ by

$$\xi \cdot (z_1, z_2, \dots) = (\xi z_1, \xi z_2, \dots)$$

This action is free on S^∞, it is in fact free when restricted to any S^{2n-1}. Since G is finite this action is also discontinuous.

It is a bit harder to get a CW-structure on S^∞/G. Start with the construction of a CW-structure on S^∞ which is isomorphically mapped to itself by the action of G.

As 0-cells take the k ~~ke~~-th roots of unity.
As 1-cells the intervalls between these in S^1. Now use the fact that $S^3 = S^1 * S^1$

join.

~~As~~ Subdivide the second S^1 factor as before. Then take as 2-cells the joins $S^1 * \{\zeta^i\}$, $i = 1, \dots, k$, as 3-cells the joins $S^1 * J_i$, $i = 1, \dots, k$, where J_i are the intervals in the second factor.

Now continue using $S^{2n+1} = S^{2n-1} * S^1$.

Then S^∞/G is a CW-complex with exactly one cell in each dimension. Call these $e_0, e_1, e_2, \dots$

Then
$$\partial_{2i}(e_{2i}) = k \cdot e_{2i-1}, \quad i = 1, 2, \dots$$
$$\partial_{2i-1}(e_{2i-1}) = 0, \quad i = 1, 2, \dots$$

So, in particular,

$$H_i(\mathbb{Z}/k\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/k\mathbb{Z} & i>0 \quad , i \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

<u>Notice</u>: For arbitrary large r there is $i \geq r$ with $H_i(\mathbb{Z}/k\mathbb{Z}) \neq 0$.

Returning to $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$

We know that $\mathrm{Aut}(F_n)$ contains finite subgroups corresponding to the isomorphism group of R_n.

None of this automorphisms is an inner automorphism, since an inner automorphism maps to the identity under the homomorphism

$$\phi_n : \mathrm{Aut}(F_n) \longrightarrow \mathrm{Aut}(F_n/F_n') = \mathrm{Aut}(\mathbb{Z}^n)$$

where F_n' is the commutator subgroup. So in particular, ϕ_n factorizes through

$$\begin{tikzcd} \mathrm{Aut}(F_n) \arrow[rr, "\phi_n"] \arrow[dr, "p_n"'] & & \mathrm{Aut}(\mathbb{Z}^n) \\ & \mathrm{Out}(F_n) \arrow[ur, "\varphi_n"'] & \end{tikzcd}$$

Since ϕ_n is injective on $\mathrm{Aut}(R_n) \subset \mathrm{Aut}(F_n)$

$P_n(\mathrm{Aut}(R_n))$ is isomorphic to $\mathrm{Aut}(R_n)$.

Remember that $\mathrm{Aut}(R_n)$ is a semi direct product of $(\mathbb{Z}/2)^n$ and S_n, the symmetric group on n elements.

As a consequence, we have

<u>6.10 Proposition:</u> If X is a connected CW-cx with $\pi_1(X) \cong \mathrm{Out}(F_n)$ or $\pi_1(X) \cong \mathrm{Aut}(F_n)$ and if the universal cover of X is contractible then $\dim X = \infty$. Here $n \geq 2$.

<u>Proof.</u> Let $H \subset \mathrm{Out}(F_n)$ be a finite cyclic subgroup (They exist) and let $X_H \xrightarrow{p_H} X$ be the covering associated to H, and $\tilde{X} \xrightarrow{p} X_H$ the universal covering of X_H (it is also the universal covering of X via $\tilde{X} \xrightarrow{p} X_H \xrightarrow{p_H} X$). Then $\tilde{X}$ is contractible and H acts freely and discontinuously on $\tilde{X}$. Therefore,

$$H_i(H) \cong H_i(\tilde{X}/H) = H_i(X_H).$$

Since $H_i(H) \neq 0$ for all $i = 2k-1$, $k \geq 1$, X_H is not finite dimensional, and thus X is not finite dimensional. $\square$