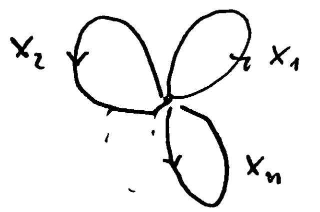
5. Automorphisms of F_n.

To find simple and\or useful generators for the group of automorphisms of F_n goes back to Nielsen (1921, 1924) and Whitehead (1935; useful).

We have already seen some automorphisms of F_n

If Δ is the graph with one vertex $*$ and n edge-pairs then $\pi_1(\Delta, *)$ is isomorphic to F_n; usually we think of F_n being free over a fixed basis $\{x_1, \ldots, x_n\}$. Then we call the graph a representation of F_n, where as before the arrows and labels indicate the isomorphism



$$F_n \longrightarrow \pi_1(\Delta, *)$$

The above graph we call the rose with n petals and denote it by R_n.

Now, any graph automorphism $f: R_n \to R_n$ induces an automorphism $f: \pi_1(R_n, *) \to \pi_1(R_n, *)$ and we already determined the structure of

$$\mathrm{Aut}\, R_n$$

which fits into an exact sequence

$$1 \longrightarrow (\mathbb{Z}/2)^n \longrightarrow \mathrm{Aut}\, R_n \underset{\sigma}{\overset{p}{\rightleftarrows}} \Sigma_n \longrightarrow 1$$

with Σ_n the symmetric group on n letters $\{x_1, \ldots, x_n\}$ and this sequence splits via σ

The subgroup $(\mathbb{Z}/2)^n$ leaves edge pairs fixed but $(\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{Z}/2)^n$ maps x_i to $\bar{x}_i$ and $\bar{x}_i$ to x_i if $\varepsilon_i = -1$ and leaves x_i and $\bar{x}_i$ fixed if $\varepsilon_i = 1$.

For a permutation g of $\{x_1, \ldots, x_n\}$

$\sigma(g)$ maps x_i to $x_{g(i)}$, $\bar{x}_i$ to $\overline{x_{g(i)}}$. And this determined also $f : \pi_1(R_n, *) \longrightarrow \pi_1(R_n, *)$ and thus we obtain an embedding

$$\mathrm{Aut}\, R_n \hookrightarrow \mathrm{Aut}\, F_n$$

The image of $\mathrm{Aut}\, R_n$ in $\mathrm{Aut}\, F_n$ is denoted by W_n reminiscent of the Weyl group in $GL_n(K)$.

But there are more automorphisms of F_n :

<u>Example</u>:

$$\begin{array}{llll} x_1 \longrightarrow x_1 x_k & , & k \neq 1 \\ x_i \longrightarrow x_i & & i \neq 1 \end{array}$$

determines a homomorphism a_{1k}. (Similar to adding the k-th row to the first in linear algebra)

$$a_{1\bar{k}} : \quad \begin{array}{l} x_1 \longrightarrow x_1 x_k^{-1} \\ x_i \longrightarrow x_i \end{array}$$

is obviously an inverse, since $a_{1\bar{k}} a_{1k}$ and $a_{1k} a_{1\bar{k}}$ keep all generators fixed.

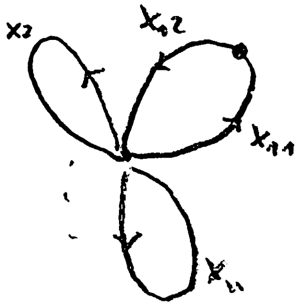
$$x_1 \xrightarrow{a_{1k}} x_1 x_k \xrightarrow{a_{1\bar{k}}} (x_1 x_k^{-1})\, x_k = x_1$$

since x_k is fixed under $a_{1\bar{k}}$

$$x_1 \xrightarrow{a_{1\bar{k}}} x_1 x_k^{-1} \xrightarrow{a_{1k}} (x_1 x_k)\, x_k^{-1} = x_1$$

$$x_i \longrightarrow x_i \quad \text{by } a_{1\bar{k}} \text{ and } a_{1k}.$$

But a_{1k} is also induced by some graph map



$$\longrightarrow \quad R_n$$

where x_{11} is mapped to x_1, x_{12} to x_k, and

$$x_i \to x_i, \quad i > 1.$$

<u>Remember</u>: The graph Γ on the left is a subdivision of R_n and there is a natural iso.

$$\pi_1(R_n, *) \longrightarrow \pi_1(\Gamma, *)$$

$$[x_1] \longmapsto [x_{11} x_{12}]$$

$$[x_i] \longmapsto [x_i]$$

so that again we have an induced automorphism

$$\pi_1(R_n, *) \longrightarrow \pi_1(R_n, *)$$

and this one is a_{1k}.

It is obvious how to generalize this to an arbitrary endomorphism of F_n. These are given by (reduced) words $w_1, \dots, w_n$ in $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ where the endomorphism is given by

$$x_i \longmapsto w_i$$

The graph Γ is then obtained from R_n by subdividing the oriented edge x_i with $|w_i|-1$ vertices into $|w_i|$ oriented edges $x_{i1}, \dots, x_{i|w_i|}$ and the isom.

$$\pi_1(R_n, *) \xrightarrow{\cong} \pi_1(\Gamma, *) \quad \text{is given by}$$

$$[x_i] \longmapsto [x_{i1} \cdots x_{i|w_i|}]$$

If $w_i = a_{i1} a_{i2} \cdots a_{i|w_i|}$ with $a_{ij} \in \{x_k^{\pm 1} : k = 1, \dots, n\}$

then $(\Gamma, *) \xrightarrow{f} (R_n, *)$ is the map

which sends x_{ij} to a_{ij}, meaning

to x_k if $a_{ij} = x_k$, to $\overline{x}_k$ if $a_{ij} = x_k^{-1}$.

Obviously:

$$\pi_1(R_n, *) \xrightarrow{\cong} \pi_1(\Gamma, *) \xrightarrow{f} \pi_1(R_n, *)$$

is then our original endomorphism.

Elementary topology tells us that f is an isomorphism if the geometric realization $G(f) : (G(\Gamma), *) \longrightarrow (GR_n, *)$ is a homotopy equivalence.

It is a little harder to show using the exercises of sheets 3, 4 and 5 that the converse also holds.

If $\alpha : G(\Gamma) \longrightarrow G(R_n)$ is a continuous map mapping $*$ to $*$ such that the induced map on fundamental groups is an isomorphism then α is a homotopy equivalence. (Recall that $G(\Gamma)$ is "naturally" homeomorphic to $G(R_n)$).

We describe now a finite family of automorphisms of F_n, called the <u>Whitehead automorphisms</u> and use our theory of foldings etc to describe an algorithm which decomposes any automorphism of F_n into a product of Whitehead automorphisms.

<u>5.1 Definition:</u> An automorphism of F_n is called a Whitehead automorphism if it is either an element of W_n or has the following form:

(Wh2) Let $a \in X \cup X^{-1}$, $X = \{x_1, \dots, x_n\}$ and $A \subset X \cup X^{-1}$ with $a \in A$, $a^{-1} \notin A$.

Define $(A,a) \in \mathrm{Aut}(F_n)$ by

$$(A,a)(x_j) = \begin{cases} x_j & \text{if } x_j = a^{\pm 1} \\ a^{\alpha_j} x_j a^{-\beta_j} & \text{if } x_j \neq a^{\pm 1} \end{cases}$$

Here $\alpha_j = \begin{cases} 1 & x_j \in A \\ 0 & x_j \notin A \end{cases}$, $\beta_j = \begin{cases} 1 & x_j^{-1} \in A \\ 0 & x_j^{-1} \notin A \end{cases}$.