4. Applications of Graph Methods to Free Group Problems

The following is an immediate Corollary to Proposition 3.21

4.1 Theorem (Howson's Thm) If S_1 and S_2 are finitely generated subgroups of a free group F. Then $S_1 \cap S_2$ is also finitely generated

Proof. Let Δ be a graph with 1 vertex, called w, such that $\pi_1(\Delta, w) \cong F$. Use 3.19 to find finite connected graphs Γ_1, Γ_2, vertices $v_1 \in V_{\Gamma_1}$, $v_2 \in V_{\Gamma_2}$ and immersions

$$f_i : (\Gamma_i, v_i) \longrightarrow (\Delta, w) \quad \text{such that}$$

$$f_i(\pi_1(\Gamma_i, v_i)) = S_i \qquad (\text{Identify } F \text{ with } \pi_1(\Delta, w))$$

Consider the pull back

$$\begin{tikzcd} \Gamma_0 \arrow[r, "g_1"] \arrow[d, "g_2"'] & \Gamma_1 \arrow[d, "f_1"] \\ \Gamma_2 \arrow[r, "f_2"'] & \Delta \end{tikzcd}$$

Then Γ_0 is finite; let v_0 be the vertex corresponding to v_1, v_2. From 3.21 we see that

$$S_1 \cap S_2 = f_1 g_1(\pi_1(\Gamma_0, v_0)).$$ So $S_1 \cap S_2$ is a free group with a finite set of generators. Then $S_1 \cap S_2$ contains a finite basis. Actually,

it is not too difficult to show the following

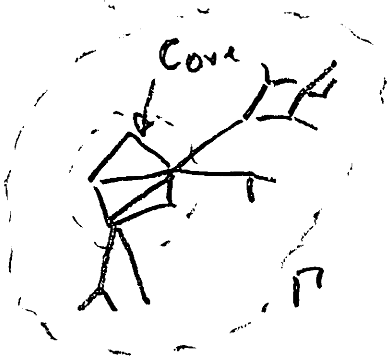
If $\{e_1, \dots, e_n\}$ is a set of generators of a free group G then G has a basis with $k \leq n$ elements. But in our situation we may use the fact that f_2 is an immersion to show that g_1 is an immersion (An exercise of this week). Thus $f_1 g_1$ is an immersion, and we obtain a basis of $S_1 \cap S_2$ by constructing a tree in the component of F_0 containing v_0 and using 2.18 $\square$

Working a little harder, we can get a bound given by Hanna Neumann (1956/57) on the rank of $S_1 \cap S_2$ which says

<u>4.2 Proposition</u> (H. Neumann) If $S_1 \cap S_2$ is not trivial then $rk(S_1 \cap S_2) - 1 \leq 2 \cdot (rk(S_1) - 1)(rk(S_2) - 1)$.

To prove 4.2 we introduce the notion of a <u>core graph</u>

Roughly, the idea of the core of a graph Γ is a subgraph C of Γ such that Γ consists of C plus some trees growing on C and C has no subcore. For finite Γ this means that C has no vertex of degree 1.



For general graphs we make the following definitions

4.3 Def.

(a) A <u>cyclically reduced loop</u> is a reduced loop $p = e_1 e_2 \cdots e_n$ with $e_n \neq \overline{e_1}$.

(b) A graph Γ is called a <u>core graph</u> if it is connected, has a least one edge and every edge belongs to some cyclically reduced loop.

(c) If Γ is a connected graph. An edge e of Γ is called <u>essential</u> if e belongs to some cycilly reduced loop.

<u>Remark.</u> If a conn. graph Γ has an essential edge then $\pi_1(\Gamma)$ is necessarily non-trivial.

(d) The core of a connected graph Γ is the sub-graph containing all essential edges and all initial vertices of essential edges.

4.4 Proposition.

(a) A <u>finite</u> connected graph is a core graph iff the valence of each vertex is at least 2

(b) If Γ is the core of a connected graph Γ' with $\pi_1(\Gamma') \neq 1$. Then Γ is a core graph.

<u>Proof.</u> Exercise. (Easy)

4.5 Definition: A subgroup H of a group G is said to satisfy the Burnside condition if for every $g \in G$ there is n s.t. $g^n \in H$.

Remark. If H satisfies the Burnside condition in G and $x \in G$ then xHx^{-1} also satisfies the Burnside condition. This follows easily from the fact that for any $g \in G$ there exists n s.t. $(x^{-1} g x)^n \in H$.

Remark: If $H < G$ has finite index, then, obviously, H satisfies the Burnside condition. In particular

4.6 If $f : \Gamma \longrightarrow \Delta$ is a finite index covering of conn. graphs, $v \in \Gamma$, then $f(\pi_1(\Gamma, v)) \subset \pi_1(\Delta, f(v))$ satisfies the Burnside condition.

More interesting is

4.7 Proposition: Let $f : \Gamma \longrightarrow \Delta$ be an immersion of connected graphs, and assume that Δ is a core-graph. Assume also that for some vertex v of Γ $f(\pi_1(\Gamma, v)) \subset \pi_1(\Delta, f(v))$ satisfies the Burnside condition.

Then f is a covering

Proof. We need to show that for a vertex $w \in \Gamma$ the restriction of f to $st(w, \Gamma) \longrightarrow st(f(w), \Delta)$ is bijective. By hypothesis it is injective. So let e be an edge of Δ with $\iota(e) = f(w)$. Since Δ is a core graph there is a cyclically reduced path p in Δ with first edge e. The Burnside condition grants a reduced loop q in Γ, starting in w and with $f(q) = p^n$ for some $n \in \mathbb{N}$. Since p is cyclically reduced, p^n is reduced. Since f is an immersion $f(q)$ is reduced. Thus $f(q) = p^n$, and the first edge of q is mapped by f to e. $\square$

Remember the algorithm 3.19 of representing a fin.-gen. subgroup of $\pi_1(\Delta, w)$ by an immersion $f : \Gamma \to \Delta$ with Γ finite? We can use 4.7 to decide whether $f(\Gamma)$ has finite index in $\pi_1(\Delta, w)$ as follows. $\pi_1(\Delta, w)$ was just a model for a free gp, so we may assume that Δ has only 1 vertex.

4.8 Remark:

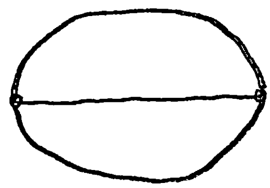
Let S be a free gp, Δ a graph with 1 vertex w with $\pi_1(\Delta, w) \cong S$ and let H be a f.g. subgp. Represent H as $f(\pi_1(\Gamma, v))$ where Γ is a finite connected graph and $f : \Gamma \to \Delta$ an immersion (possible by 3.19) Δ is a core graph. If f is a covering, then obviously, $f(\pi_1(\Gamma, v))$ has finite index in $\pi_1(\Delta, w)$. If f is not a covering, then by 4.7 $f(\pi_1(\Gamma, v))$ does not satisfy the Burnside condition, and thus $f(\pi_1(\Gamma, v))$ must have infinite index.

4.9 We finally come to the proof of the Hanna Neumann bound on $\mathrm{rk}(S_1 \cap S_2)$ in Howson's thm (4.1)

$$(*) \quad \mathrm{rk}(S_1 \cap S_2) \leq 2(\mathrm{rk}\, S_1 - 1)(\mathrm{rk}\, S_2 - 1) \quad , \text{ if}$$

S_1, S_2 are f.g. subgroups of a free group

First notice that $S_1 \cup S_2$ is then contained in a subgroup of finite rank in the free group. So it suffices to assume this. We now that F_2 contains subgroups of any finite rank. So we may assume that $S_1, S_2 \subset F_2$. We represent F_2 as the fundamental group of Δ:



Represent S_1, S_2 by immersions $f_i : \Gamma_i \to \Delta$ and $S_1 \cap S_2$ as the fundamental group of a component Γ_3 of the pull-back of

$$\begin{array}{ccc} & & \Gamma_1 \\ & & \downarrow{\scriptstyle f_1} \\ \Gamma_2 & \xrightarrow{f_2} & \Delta \end{array}$$

As before Γ_1, Γ_2 are finite graphs, so that Γ_3 is finite. We may assume that the basepoint v_3 for $\pi_1(\Gamma_3)$ lies in the core Γ_3' of Γ_3. If not, conjugating everything inside Δ will do this without affecting the ranks of the subgroups involved. Replacing Γ_1, Γ_2 by their core graphs we obtain a comm. diagram

$$\begin{tikzcd} \Gamma_3' \arrow[r, "g_1"] \arrow[d, "g_2"'] & \Gamma_1' \arrow[d, "f_1'"] \\ \Gamma_2' \arrow[r, "f_2'"'] & \Delta \end{tikzcd}$$

where f_i' are immersions and Γ_3' is a (connected) subgraph of the pullback

Looking at Δ we see that the valence of each vertex of each $\Gamma_1', \Gamma_2', \Gamma_3'$ is either 2 or 3.

Removing vertices of valence 2 will not change the fundamental groups (Exercise of some early homework sheet).

Let Θ be any connected graph. Then $\#\text{edge pairs}_\Theta - \#\text{vertices}_\Theta = rk(\pi_1 \Theta) - 1$

since $\#\text{edge pairs}_T - \#\text{vertices}_T = -1$ for any tree T.

Now let all vertices of Θ have valence 3.

Then $\#\text{edge pairs}_\Theta = \frac{3}{2}\#\text{vertices}_\Theta$

Thus $rk(\pi_1 \Theta) - 1 = (\frac{3}{2} - 1)\#\text{vertices}_\Theta$ i.e.

$\#\text{vertices}_\Theta = 2(rk(\pi_1(\Theta)) - 1)$

But because of the pull back situation we have

$\#\text{vertices}_{\Gamma_3'} \leq \#\text{vertices}_{\Gamma_1'} \cdot \#\text{vertices}_{\Gamma_2'}$.

and any vertex of Γ_3' of valence 2 is mapped to a vertex of Γ_i' of valence at least 2.

Thus, if Γ_i'' is Γ_i' with all vertices of valence 2 removed we have

$$2(\mathrm{rk}(S_1 \cap S_2)-1) = \#\,\mathrm{vertices}_{\Gamma_3''} \leq \#\,\mathrm{vertices}_{\Gamma_1''} \cdot \#\,\mathrm{vertices}_{\Gamma_2''}$$

$$= 4\,(\mathrm{rk}(S_1)-1)(\mathrm{rk}(S_2)-1). \qquad \square$$

<u>Comment</u>: Hanna Neumann conjectured that in (*) the factor 2 could be replaced by 1. This Hanna Neumann Conjecture from 1957 created quite some activity, including some claims of proofs which turned out to be incomplete.
It finally was proved last year in June, first by

Joel Friedman (arXiv 2011) and a few days later by

Igor Mineyev (Annals of Math. 175 (2012), 22 pages)

Annals of Math is probably <u>the</u> most prestigeous journal for pure mathematics.