Notice that the condition $e_1 \neq \bar{e}_2$ means that we can choose orientations on Γ and  so that the induced orientations on  agree, i.e. the condition implies that the push-out exists

<u>3.4 Remark</u> If $f : \Gamma \to \Delta$ is a graph map and $e_1, e_2 \in E_\Gamma$ with $\iota(e_1) = \iota(e_2)$ and $f(e_1) = f(e_2)$, then (e_1, e_2) is admissible (were $\bar{e}_2 = e_1$ then $\overline{f(e_2)} = f(\bar{e}_2) = f(e_1) = f(e_2)$; but this contradicts $\overline{f(e_2)} \neq f(e_2)$). Furthermore, there exists a unique map $\Gamma/_{[e_1 = e_2]} \xrightarrow{f'} \Delta$ s.t.

$f' \circ p = f$ (by the push-out property).

This leads to

<u>3.5 Remark</u> If Γ is finite and $f : \Gamma \longrightarrow \Delta$ a graph map. Then there is a sequence of foldings

$\Gamma = \Gamma_0 \xrightarrow{p_1} \Gamma_1 \xrightarrow{p_2} \cdots \xrightarrow{p_n} \Gamma_n$ and a

unique immersion $\Gamma_n \xrightarrow{f'} \Delta$ such that

$f = f' \circ p_n \circ \cdots \circ p_1$.

<u>Proof</u>. By induction on the number of edges of Γ. If there is at most one edge in Γ there is nothing to prove.

Let Γ have $n > 1$ edges. If there is no adm. pair, then there can be no folding and also f is an immersion uniquely determined by f.

If there is no admissible pair (e_1, e_2) with $f(e_1) = f(e_2)$ then again f is an immersion and there is no folding $\Gamma \xrightarrow{p} \Gamma/_{[e_1=e_2]}$ such that f factors through $\Gamma \xrightarrow{p} \Gamma/_{[e_1=e_2]}$.

Finally, if there exists an admissible pair (e_1, e_2) such that f factors through $\Gamma \xrightarrow{p} \Gamma/_{[e_1=e_2]}$ then $\Gamma/_{[e_1=e_2]}$ has one edge less than Γ and there exists a unique $f_1 : \Gamma/_{[e_1=e_2]} \to \Delta$ with $f = f_1 \circ p$. $\square$

Coverings in graphs have the same properties as coverings in the category of topological spaces.

<u>3.6 Proposition:</u> (Path Lifting)

Let $\Gamma \xrightarrow{f} \Delta$ be a covering, p a path in Δ and v a vertex of Γ with $f(v) = \iota(p)$. Then there exists a unique path $\tilde{p}$ in Γ starting in v with $f(\tilde{p}) = p$.

<u>Proof.</u> Induction on $|p|$. If $|p| = 0$, this is obvious. If $|p| = n > 0$ and the result holds for paths of length $\leq n-1$, write

$p = p_1 e$, we have a unique lift $\tilde{p}_1$ starting in v. If it ends in v_1 then $f(v_1) = \iota(e)$. Since f is a covering there is a <u>unique</u> $\tilde{e} \in st(v_1, \Gamma)$ with $f(\tilde{e}) = e$. Then $\tilde{p} = \tilde{p}_1 \tilde{e}$ is the unique path with $\iota\tilde{p} = v$ and $f\tilde{p} = p$. ☐

<u>3.7 Proposition</u> (Homotopy lifting for paths).

$\Gamma \xrightarrow{f} \Delta$ a covering, $p_1 \simeq p_2$ homotopic paths in Δ, $v \in \Gamma$ with $f(v) = \iota(p_1)$. Then $\tilde{p}_1 \simeq \tilde{p}_2$.

<u>Proof</u> If $e\bar{e}$ is a 1-step backtrack in Δ, $\tilde{q}$ a path in Γ with $f\tilde{q} = e\bar{e}$ then $\tilde{q}$ is a 1-step backtrack; since a lift $\tilde{e}$ of e implies that $\tilde{e}\,\bar{\tilde{e}}$ is a lift of $e\bar{e}$. By uniqueness of path liftings $\tilde{q} = \tilde{e}\,\bar{\tilde{e}}$. ☐

<u>3.8 Proposition</u> (Lifting of maps)

Let $f : \Gamma \longrightarrow \Delta$ be a covering, Θ a connected graph, $g : \Theta \longrightarrow \Delta$ a graph map and $w \in \Theta$, $v \in \Gamma$ vertices with $f(v) = g(w)$.

Then there exists a lift $\tilde{g} : \Theta \longrightarrow \Gamma$ of g with $\tilde{g}(w) = v$ (i.e. a graph map $\tilde{g} : \Theta \to \Gamma$ s.t. $f \circ \tilde{g} = g$) iff

$f\pi_1(\Gamma, v) \supset g(\pi_1(\Theta, w))$ in $\pi_1(\Delta, f(v))$

Furthermore, $\tilde{g}$ is unique

Uniqueness is clear since Θ is connected:

If w' is a vertex of Θ, p a path from w to w' then there is a unique lift of gp starting in v. Its endpoint then must be $\tilde{g}(w')$.

If e is an edge of Θ, p a path from w to $\iota(e)$. Then there is a unique $\tilde{e}$ in Γ with $f(\hat{e}) = g(e)$ and $\iota(\hat{e}) = \tau(\tilde{p})$. Thus necessarily $\tilde{g}(e) = \hat{e}$.

Existence: For any $w' \in V_\Theta$ choose some path p' from w to w' and let $\tilde{p}$ be the unique lift of p' with $\iota\tilde{p}' = v$. Define $\tilde{g}(w') = \tau\widetilde{p'}$.

For any edge $e \in E_\Theta$ choose some path p from w to ιe and define $\tilde{g}(e)$ as the unique edge in Γ with $\iota(\tilde{g}(e)) = \tau\tilde{p}$ and $f(\tilde{g}(e)) = g(e)$

We have to show that the map $\tilde{g}$ is well-defined and a graph homom.

So let p_1 and p_2 be two paths from w to w'. Then $p_1\overline{p_2}$ is a loop in Θ

starting in w. Thus $g(p_1\bar{p}_2)$ is a loop in Δ starting in $f(v)$. By hypothesis there exists a loop $\hat{q}$ in Γ starting in v such that

$q = f\hat{q}$ is homotopic to $g(p_1\bar{p}_2) = gp_1\, g\bar{p}_2$

$$= gp_1\, \overline{gp_2}$$

Therefore $\widetilde{gp_1\,\overline{gp_2}}$ is homotopic to $\hat{q}$ and in particular a loop $\Rightarrow$ $\tau\, \widetilde{gp_1} = \tau\, \widetilde{gp_2}$.

Therefore $\tilde{g}$ is well-defined.

$\tilde{g}(\iota e) = \iota(\tilde{g}(e))$ is clear by construction and the fact that $\tilde{g}$ is well-defined.

$\tilde{g}(\bar{e})$ is obtained by choosing a path from w to $\iota(\bar{e})$. We do this by first choosing a path from w to $\iota(e)$ and concatenate with e. Then $\tilde{g}(\bar{e}) = \overline{\tilde{g}(e)}$ is clear.

Existence of the desired lifting obviously implies the condition $g(\pi_1(\Theta, w)) \subset f(\pi_1(\Gamma, v))$.

$\square$

<u>3.9.(1) Proposition</u> (Injectivity)

If $f: \Gamma \to \Delta$ is a covering, then for any $v \in \Gamma$ the map $f: \pi_1(\Gamma, v) \to \pi_1(\Delta, f(v))$ is injective. (This follows immediately from 3.7)