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Free Groups and Graphs

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Homework 7 — Due: December 3, 2012

Problem 1

We know that the fundamental group $\pi_1(\Gamma, v)$ of a pointed graph (Γ, v) is a free group and that all subgroups of free groups are free. A subgroup $F \leq \pi_1(\Gamma, v)$ is called a *free factor* of $\pi_1(\Gamma, v)$ if any free basis of F can be extended to a free basis of $\pi_1(\Gamma, v)$.

Let $inc\colon \Gamma_1 \to \Gamma$ be the inclusion of a subgraph and $v \in V(\Gamma_1)$. Show that $\pi_1(inc)\colon \pi_1(\Gamma_1, v) \to \pi_1(\Gamma, v)$ is the inclusion of a free factor of $\pi_1(\Gamma, v)$.

Problem 2

Let $F_2 = \langle a, b\rangle$ be a free group on 2 generators and $\phi \in Aut(F_2)$ an automorphism of F_2 such that $\phi(\langle a \rangle) \subseteq \langle a \rangle$.

Show that in fact $\phi(\langle a \rangle) = \langle a \rangle$ holds. (Hint: Taking the abelianization of a group is functorial and an automorphism of a group induces an automorphism of its abelianization.)

Problem 3

Let

$$\begin{array}{ccc} \Gamma & \xrightarrow{F} & \Delta \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle p} \\ \Sigma & \xrightarrow[f]{} & \Theta \end{array}$$

be a pullback diagram of graphs. Show that if p is an immersion then so is P. Also show that if p is a covering then so is P.

Problem 4

Let Γ be a connected graph and T a tree, and let $p\colon T \to \Gamma$ be a finite-sheeted covering (i.e. the fibers of p are finite sets). Show that p is in fact an isomorphism.