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Free Groups and Graphs

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Homework 4 Due: November 12, 2012

Problem 1

A graph Γ is called *star-shaped* if there exists a vertex $v_0 \in V = V(\Gamma)$ and an orientation $\mathcal{O} \subset E(\Gamma)$ such that

(i) for all $e \in \mathcal{O}$ we have $\iota(e) = v_0$; and

(ii) the map $\mathcal{O} \to V,\ e \to \tau(e)$ is a bijection onto $V \setminus \{v_0\}$.

Denote by $|V|$ the cardinality of the vertex set V and by $G(\Gamma)$ the geometric realization of the graph Γ. Prove the following assertions:

$a)$ Whenever $|V| \leq |\mathbb{R}|$ we can find an injective continuous map $G(\Gamma) \to \mathbb{R}^2$.

$b)$ If V is infinite there cannot exist an embedding $G(\Gamma) \to \mathbb{R}^n$ for any n.

Problem 2

A graph is *locally finite* if the star of every vertex is finite. Show that for any locally finite graph Γ with countable vertex set $V(\Gamma)$ there always exists a closed embedding $G(\Gamma) \to \mathbb{R}^3$ (images of closed sets are closed).

Problem 3

Let Γ be a graph and $\Gamma^{(2)}$ its second subdivision as defined in Homework 3. Note that in contrast to $G(\Gamma)$, the geometric realization $G\left(\Gamma^{(2)}\right)$ is always a simplicial complex (i.e. for each edge $e \in E\left(\Gamma^{(2)}\right)$ the vertices $\iota(e)$ and $\tau(e)$ are distinct, and between two vertices of $\Gamma^{(2)}$ there is at most one edge).

Moreover, let $f\colon [0,1] \to G\left(\Gamma^{(2)}\right)$ be a continuous map for which $f(0)$ and $f(1)$ lie in $V\left(\Gamma^{(2)}\right)$, and assume there exist $0 = a_0 < ... < a_n = 1$ such that for all $i = 0, ..., n$ there is a vertex $v_i \in V\left(\Gamma^{(2)}\right)$ with $f(st(a_i)) \subseteq st(v_i)$.

For each i choose a vertex v_i such that $f(st(a_i)) \subseteq st(v_i)$ and define a continuous map $g\colon [0,1] \to G\left(\Gamma^{(2)}\right)$ by

(i) each a_i gets mapped to v_i;

(ii) the interval $[a_{i-1}, a_i]$ gets mapped either to the vertex a_i if $v_{i-1} = v_i$ or linearly to the unique edge connecting v_{i-1} to v_i if $v_{i-1} \neq v_i$.

Show that f and g are homotopic as paths, i.e. relative to their endpoints.

Remark. Note that g can be thought of as the geometric realization of a combinatorial edge path and convince yourself that a subdivision of the interval $0 = a_0 < ... < a_n = 1$ with the above mentioned property concerning stars of vertices always exists. Hence, every continuous path in $G\left(\Gamma^{(2)}\right)$ is homotopic as a path to the geometric realization of a combinatorial edge path.