Freie Universität Berlin
Institut für Mathematik

Prof. Dr. Elmar Vogt
Sebastian Meinert

Free Groups and Graphs

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Homework 13 Due: January 28, 2013

Denote by A_m the graph with $m+1$ vertices and $2m$ edges whose geometric realization is an interval. Denote by C_m the quotient graph $A_m/(v_0 \sim v_m)$, where v_0 and v_m are the two vertices of A_m of valence one. Observe that the geometric realization of C_m is the circle.

Let Γ be a finite connected graph. A *free loop* of length m in Γ is a graph homomorphism $C_m \to \Gamma$. Two free loops $l_1 \colon C_{m_1} \to \Gamma$ and $l_2 \colon C_{m_2} \to \Gamma$ are *homotopic* if their geometric realizations $G(l_1), G(l_2) \colon S^1 \to G(\Gamma)$ are (freely) homotopic as maps between topological space. Define $\Lambda(\Gamma)$ as the set of free loops in Γ modulo homotopy.

Problem 1

Let Γ be a finite connected graph all of whose vertices are of valence at least three. Every graph endomorphism $f \colon \Gamma \to \Gamma$ induces a map $\Lambda(f) \colon \Lambda(\Gamma) \to \Lambda(\Gamma)$ which takes $[l]$ to $[f \circ l]$.

Show that if $\Lambda(f) = id_{\Lambda(\Gamma)}$ then $f = id_\Gamma$.

Problem 2

Denote by Out_n the Outer space in rank n. Let $[\Gamma, g] \in Out_n$, where Γ is a metric graph all of whose vertices are of valence at least three and $g \colon R_n \to \Gamma$ is a homotopy equivalence, the *marking*. Recall that the outer automorphism group $Out(F_n)$ acts on Out_n in the following way: Every $[\phi] \in Out(F_n)$ corresponds to the free homotopy class of a homotopy equivalence $\phi \colon R_n \xrightarrow{\simeq} R_n$, and the group action takes $([\phi], [\Gamma, g])$ to $[\Gamma, g \circ \phi]$.

Show that the stabilizer of $[\Gamma, g]$ under this action is isomorphic to $Isom(\Gamma)$, the isometry group of Γ. (Hint: At a certain point you might need to make use of Problem 1.)

Problem 3

For $[\Gamma, g] \in Out_n$ denote by $\sigma([\Gamma, g])$ the set

$$\{[\Gamma', g'] \in Out_n \mid \exists \text{ homeomorphism } h\colon \Gamma \to \Gamma' \text{ s.th. } g' \simeq h \circ g\},$$

the *open simplex* spanned by $[\Gamma, g]$. The elements of $\sigma([\Gamma, g])$ can be obtained from $[\Gamma, g]$ by varying the edge lengths of Γ while keeping them positive.

Show that there is a bijection from $\sigma([\Gamma, g])$ to the standard $(k-1)$-simplex

$$\left\{(x_1, ..., x_k) \in \mathbb{R}^k \;\middle|\; \sum_{i=1}^{k} x_i = 1,\ x_i > 0 \text{ for all } i\right\},$$

where k is the number of edges of Γ. Pay particular care to showing that your map is well-defined (or, depending on the direction of your map, injective).