Whitehead automorphisms:

(a) W_n (Weyl group of $\mathrm{Aut}\, F_n$)

(b) $a \in X \uplus X^{-1}$ $\quad X = \{x_1, \dots, x_n\}$ basis of F_n

$A \subset X \cup X^{-1}$

with $a \in A$ $\quad a^{-1} \notin A$

$$(A, a)(x_j) = \begin{cases} x_j & x_j = a^{\pm 1} \\ a^{\alpha_j} x_j a^{-\beta_j} & x_j \neq a^{\pm 1} \end{cases}$$

$$\alpha_j = \begin{cases} 1 & x_j \in A \\ 0 & x_j \notin A \end{cases} \qquad \beta_j = \begin{cases} 1 & x_j^{-1} \in A \\ 0 & x_j^{-1} \notin A \end{cases}$$

Claim: Whitehead automorphisms generate $\mathrm{Aut}\, F_n$.

We will actually give an algorithm for writing a given automorphism of F_n as a product of Whitehead automorphisms

Set up to graphically describe an automorphism:

As before we call R_n the graph with 1 vertex and n edge pairs with a choice $\mathcal{v}$ of orientation and a labeling $x_1, \dots, x_n$ of the edges of $\mathcal{v}$. There is a natural identification of $\pi_1(R_n, *)$ with $F(x_1, x_2, \dots, x_n)$.

For today's lecture a <u>homotopy equivalence between two connected graphs</u> Γ, Δ is a graph map f such that f induces an isomorphism $f: \pi_1(\Gamma, v) \to \pi_1(\Delta, f(v))$ for some vertex v (and therefore all vertices) of Γ.

(Compare the remarks at bottom of page 5.4 and top of 5.5)

Given a graph Γ and a homotopy equivalence $f: \Gamma \to R_n$, we know that for any vertex v of Γ $\pi_1(\Gamma, v)$ is isomorphic to $F(x_1, \dots, x_n)$ $(=: F_n)$. Once we make this isomorphism explicit f induces a well defined automorphism of F_n. (Compare for a particular example page 5.4)

Remember from chapter 2 (see 2.18) that we get an explicit isomorphism $\pi_1(\Gamma, v) \to F_n$ by

(*) choosing
- a vertex (basepoint) v_0 of Γ
- a maximal tree T in Γ
- an orientation of the edges in $\Gamma \setminus T$
- a labeling $e_1, \dots, e_n$ of these edges.

Γ is here as always connected.

The isomorphism then maps $\pi_1(\Gamma, v_0)$ to F_n by mapping the reduced path

$[v_0, \iota(e_i)]_T \, e_i \, [\tau(e_i), v_0]_T$ to the reduced word x_i,

Given in addition the homotopy equivalence the associated automorphism is then

$$[x_i] \to \left[f[v_0, \iota(e_i)]_T \, f(e_i) \, f[\tau(e_i), v_0]_T \right]$$

$$\in \pi_1(R_n, *) = F_n$$

Therefore we make the following

<u>5.2 Definition.</u> A (based) marking of a connected finite graph Γ is a choice (*). A graph with a marking and homotopy equivalence $f: \Gamma \to R_n$ we call a marked graph

<u>Remark.</u> Later we also consider markings without base points. That is the reason for thinking of our markings as based markings

The upshot of all this is

5.3 A marked graph $\Gamma_M = \{\Gamma, f, b, T, e_1, \dots, e_n\}$ determines a well defined automorphism of F_n and any automorphism of F_n is determined by a marked graph (Compare page 5.4)

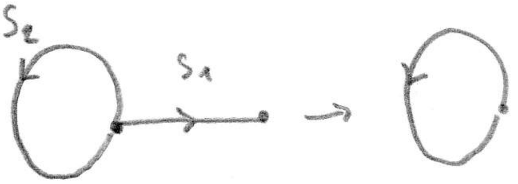
Since Γ is finite by 3.5 there is a sequence $p_1, \dots, p_r$ of foldings $\Gamma_1 = \Gamma \xrightarrow{p_1} \Gamma_2 \to \cdots \xrightarrow{p_r} \Gamma_0$ and an immersion $\Gamma_0 \xrightarrow{g} R_n$ such that

$f = g \circ p_r \circ \cdots \circ p_1$

We know that the p_i are surjective on π_1. Since f is a homotopy equivalence, all p_i must also be homotopy equivalences, consequently, also g is a homotopy equivalence.

We have investigated the effect of foldings on π_1 (see 3.20)
If the folding is a homotopy equivalence, there are 2 possibilities pictured in the graph to be folded.

with $\tau(s_1) \neq \tau(s_2)$



with $\tau(s_1) \neq \iota(s_1)$

we will analyse what projections do with respect to automorphisms later.

For immersions we have:

<u>5.4 Proposition.</u> Let $f : \Gamma \longrightarrow R_n$ be a homotopy equivalence and immersion with Γ finite. Then f is an isomorphism. In particular Γ has only one vertex and any autom. of R_n given by any marking of Γ is an element of W_n.

<u>Proof</u> By 4.10 we can add edges to Γ and extend f to obtain a covering $f' : \Gamma' \longrightarrow R_n$

Clearly $f'(\pi_1(\Gamma')) \supset f(\pi_1(\Gamma)) = \pi_1(R_n)$

Since f' is a covering this means that $f' : \pi_1(\Gamma') \xrightarrow{\cong} \pi_1(R_n)$. But $f : \pi_1(\Gamma) \xrightarrow{\cong} \pi_1(R_n)$

So we cannot add any edges, i.e. $\Gamma' = \Gamma$, $f' = f$;
Thus f is a degree 1 covering i.e. an isomorphism. $\square$

So it remains to analyze what happens at foldings.

~~and~~