3. Graphs II

Coverings and Immersions

3.1 Definition: If v is a vertex of the graph Γ then the star of v in Γ is the set of edges

$$St(v,\Gamma) = \{e \in E : \iota(e) = v\}$$

The cardinality of of $St(v,\Gamma)$ is called the valence of v in Γ.

To avoid too much formalism in describing examples let us briefly describe the geometric realization of a graph Γ (compare 2.11)

This is a topological space with additional structure (1-dim. CW-complex). We will often use the same notation for Γ and its realization. I hope this does not lead to any confusion. For the moment, denote it by $G(\Gamma)$. As a space

$$G(\Gamma) = V_\Gamma \sqcup \bigsqcup_{e \in \mathcal{O}} [0,1]_e \; / \sim$$

where $\mathcal{O}$ is an orientation of Γ and $[0,1]_e$ is a copy of $[0,1]$. $\sim$ is the equivalence relation which identifies $0 \in [0,1]_e$ (denote it by 0_e) with $\iota(e)$) and 1_e with $\tau(e)$. Now give $G(\Gamma)$ the quotient topology, where V_Γ is discrete, and $[0,1]$ has its usual topology.

There are in principle two ways to incorporate the CW-structure into the space $G(\Gamma)$.

We always have a filtration $V_\Gamma \subset G(\Gamma)$ with V_Γ a discrete subset.

in the first method we demand that up to homeomorphism $G(\Gamma)$ is constructed from V_Γ as described above

via maps $f_e : \{0,1\} \longrightarrow V_\Gamma$, $e \in E$, E some set : i.e.

$$G(\Gamma) = V_\Gamma \sqcup \coprod_{e \in E} [0,1]_e \,/\sim$$

$$0_e \sim f_e(0), \quad 1_e \sim f_e(1).$$

in the second method one demands that

$$G(\Gamma) \setminus V_\Gamma \cong \coprod_{e \in E} (0,1)_e$$

and $\overline{(0,1)_e} = (0,1)_e \cup$ finite set of vertices in V_Γ.

and $A \subset G(\Gamma)$ is closed iff $A \cap (0,1)_e$ is closed in $(0,1)_e$.

One calls any space of the form $G(\Gamma)$ a 1-dimensional CW-complex.

To any 1-dimensional CW-complex given as

$$V \sqcup \coprod_{a \in A} [0,1]_a \,/\sim \qquad V \text{ discrete, } A \text{ set}$$

$f_a : \{0,1\} \longrightarrow V$, $a \in A$

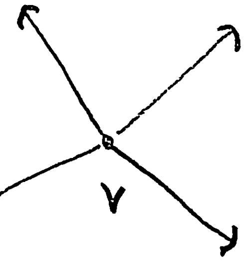
we obtain a graph Γ as follows:

$$V_\Gamma = V, \qquad v = \{e_a ; a \in A\} \quad \iota(e_a) = f_a(0),$$
$$\tau(e_a) = f_a(1)$$
$$E = v \sqcup \bar{v} .$$

We call a 1-dimensional CW-cx given by

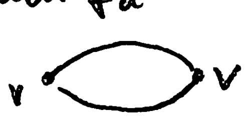
$\{V, \ f_a : \{0,1\} \longrightarrow V, a \in A\}$ a <u>simplicial</u> complex (of dimension 1) if each f_a is injective and $\operatorname{im} f_a = \operatorname{im} f_{a'}$ implies $a = a'$.

If Γ is a graph such that $G(\Gamma)$ is a simplicial complex (In some books on graph theory, every graph is of this type), then we can visualize the star of a vertex v like an open star, where we usually add v to the star. Then each star is contractible to v.



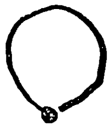
← endpoints are missing

we get the same picture if we only require that each f_a is injective



star v =

The only difference shows up if we have loop edges



v

then after adding v star v is again

v

Further useful notions.

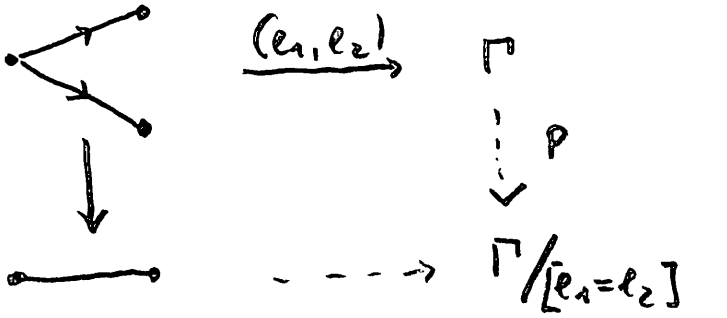
3.2 Definition: A map $f : \Gamma \longrightarrow \Delta$ of graphs induces for each $v \in V_\Gamma$ a map

$$f_v : St(v, \Gamma) \longrightarrow St(f(v), \Delta)$$

f is called an <u>immersion</u> if f_v is injective for all v. If f_v is surjective for all v we call f <u>locally surjective</u>, if each f_v is bijective we call f a <u>covering</u>.

3.3 Definition: A <u>pair</u> (e_1, e_2) <u>of edges</u> of Γ is called <u>admissible</u> if $\iota(e_1) = \iota(e_2)$ and $e_2 \neq \overline{e}_1$. The we can identify $\tau(e_1) = \tau(e_2)$, $e_1 = e_2$, $\overline{e}_1 = \overline{e}_2$ to obtain a new graph denoted by $\Gamma/_{[e_1 = e_2]}$

i.e. we form the push-out



The map p is called a folding of e_1 and e_2 in Γ.