Graphs I
Basic Notions

We follow practically verbatim:

J. Stallings: Topology of finite graphs, Invent. math. 71, 551–565 (1983)

2.1 Definition. A graph Γ consists of two sets E and V and two maps $\bar{\ }: E \longrightarrow E$, $e \longmapsto \bar{e}$ and $\iota: E \longrightarrow V$

satisfying the following rules

(i) $\bar{\bar{e}} = e$ (ii) $e \neq \bar{e}$

2.2 Notations:
An element $e \in E$ is called a directed edge of Γ and $\bar{e}$ is called the reverse of e; $\iota(e)$ is called the initial vertex of e. We define the terminal vertex $\tau(e)$ to be $\iota(\bar{e})$, i.e. the initial vertex of the reverse of e

An orientation of Γ is the choice of exactly one edge in each pair $\{e, \bar{e}\}$. Elements of V are called vertices of Γ

2.3 Definition. A map $f: \Gamma \to \Delta$ between graphs consists of two maps $f_E: E_\Gamma \longrightarrow E_\Delta$, $f_V: V_\Gamma \longrightarrow V_\Delta$ (where for a graph Ψ E_Ψ and V_Ψ are the edges and vertices of Ψ) which are compatible with the structure maps $\bar{\ }$ and ι, i.e.

$$f_E(\bar{e}) = \overline{f_E(e)}, \quad f_V(\iota(e)) = \iota f_E(e).$$

Usually we drop the indices E, V from our notation since usually it is clear what f is applied to.

Clearly, compositions of graph maps are graph maps

$$\left(g\circ f(\bar{e}) = g(\overline{f(e)}) = \overline{g\circ f(e)};\ g\circ f(\iota(e)) = g(\iota(f(e))) = \iota(g\circ f(e)) \right)$$

and we have a category with graphs as objects and graph maps as morphism.

There are a number of categorical concepts one might (or might not) want to study for graphs. Most are quite obvious. We will introduce these concepts when we come across them later on. But two concepts appear frequently. We discuss them now.

<u>2.4</u> <u>Definitions</u>: Let $\mathcal{C}$ be a category. Given morphisms

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \\ C & \end{tikzcd}$$

in $\mathcal{C}$. The <u>push-out</u> of this diagram is an object D in $\mathcal{C}$ with morphisms $b : B \longrightarrow D$ and $c : C \longrightarrow D$ s.t.

(i)
$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d, "b"] \\ C \arrow[r, "c"] & D \end{tikzcd}$$
commutes and

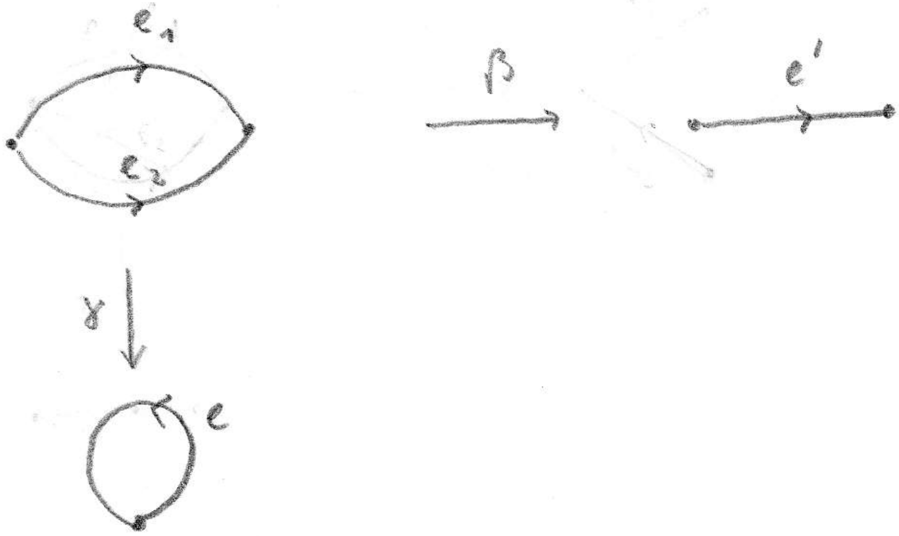
(ii) For any pair of morphisms $c' : C \longrightarrow D'$, $b' : B \longrightarrow D'$ in $\mathcal{C}$ with $b'\beta = c'\gamma$ there exists a unique morphism $h : D \longrightarrow D'$ such that $h\circ b = b'$, $h\circ c = c'$.

i.e

$$\begin{tikzcd}
A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d, "b"] \arrow[ddr, bend left, "b'"] & \\
C \arrow[r, "c"] \arrow[drr, "c'"'] & D \arrow[dr, dashed, "h"', "\exists!"] & \\
& & D'
\end{tikzcd}$$

As before, the drawn out diagram is given and commutes. Then we find a unique dotted arrow h making everything commute.

Push-Outs need not exist in every category. For example consider in the category of graphs:



where β is the obvious map, $\gamma(e_1) = e$, $\gamma(e_2) = \bar{e}$

(We draw for each pair $\{d, \bar{d}\}$ of edges only one, the map ι is indicated by the arrows)

Assume the pushout graph Γ, b, c exists. Then

$c(e) = c \circ \gamma(e_1) = b \circ \beta(e_1) = b(e')$

$c(\bar{e}) = c \circ \gamma(e_2) = b \circ \beta(e_2) = b(e')$. i.e. if c is a graph map we have $\overline{c(e)} = c(\bar{e}) = c(e)$ contradicting 2.1.(iii)

But there is a mild condition on the maps f_1, f_2 in the diagram in the category of graphs

$$\begin{tikzcd} \Gamma \arrow[r, "f_1"] \arrow[d, "f_2"'] & \Delta_1 \\ \Delta_2 & \end{tikzcd}$$

which guarantees the existence of a pushout $g_i : \Delta_i \to \Delta$, $i = 1, 2$.

<u>2.5 Claim</u>: If we can orient Γ, Δ_1, Δ_2 such that f_1 and f_2 preserve orientations then the above diagram admits a pushout.

<u>Proof</u>. Let E_i, V_i be the oriented edges and the vertices of Δ_i

Let V, $g_{V_i} : V_i \to V$ be the push-out of

$$\begin{tikzcd} V_\Gamma \arrow[r, "f_1"] \arrow[d, "f_2"'] & V_1 \arrow[d, "g_{V_1}"] \\ V_2 \arrow[r, "g_{V_2}"'] & V \end{tikzcd}$$

i.e. $V = V_1 \sqcup V_2 / \sim$ where $\sim$ is generated by:

$v_1 \in V_1$, $v_2 \in V_2$ are equivalent if there is $v \in V_\Gamma$ such that $f_i(v) = v_i$, and define $g_{V_i}(w_i) = [w_i]$, $i = 1, 2$, $w_i \in V_i$.

2.6 Exercise: Check that this is in fact a push out in the category of sets.

Similarly we let E be the push-out of the diagram

$$\begin{tikzcd} E_\Gamma \arrow[r] \arrow[d] & E_1 \arrow[d] \\ E_2 \arrow[r] & E \end{tikzcd}$$

We define $\bar{\ } : E \to E$ via representatives, i.e.

$\overline{[e_i]} = [\bar{e}_i]$.

and $\iota : E \to V$ by $\iota[e_i] = [\iota e_i]$.

We have seen in the example, that in general this leads to maps $\bar{\ }$ which have fixed elements. That $(E, V, \bar{\ }, \iota)$ is in fact a graph is a consequence of the orientation conditions in the Claim.

First note that an orientation of (the) target of a graph map induces a unique orientation of the source which makes the map orientation preserving. Thus, our condition says that there are orientations $\mathcal{O}_1$ of Δ_1 and $\mathcal{O}_2$ of Δ_2 such that the induced orientations on Γ are the same. This implies that an $e_i \in E_1 \sqcup E_2$ with $e_i \in \mathcal{O}_1 \cup \mathcal{O}_2$ can be equivalent to $e_j \in E_1 \sqcup E_2$ only if e_j is also in $\mathcal{O}_1 \cup \mathcal{O}_2$. This implies, in particular, that e_i and $\bar{e}_i$ are never equivalent, i.e

$\overline{[e_i]} \neq [e_i]$. $\square$