For later use, we want to define a natural map

$$h_n : \pi_n(X,A) \longrightarrow H_n(X,A)$$

(which also works in the absolute case, with a mild modification for $n=0$)

Let $f : (D^n, S^{n-1}) \longrightarrow (X,A)$ represent $\alpha \in \pi_n(X,A)$

Let $\iota_n$ be the standard generator of $H_n(D^n, S^{n-1})$

(see remark on page 5.13)

Then define

$$h_n[f] = H_n(f)(\iota_n) \in H_n(X,A)$$

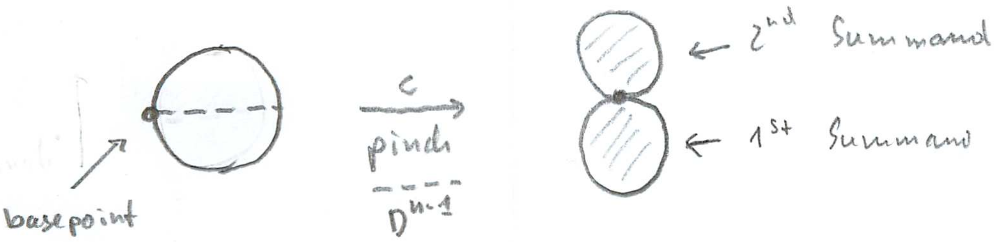
This map is obviously natural. Furthermore, in the Hurewicz theorem this map is obviously the isomorphism $\pi_i(X,A) \longrightarrow H_i(X,A)$ for $i<n$, and also our isomorphism for $i=n$, using the cellular complex for computing $H_i(X,A)$.

A little more can be said about $h_n$, even if $(X,A)$ is not $(n-1)$-connected.

<u>15.10</u> Proposition. If $n>1$, then $h_n : \pi_n(X,A) \longrightarrow H_n(X,A)$ is a homomorphism.

To see this consider the comultiplication map

$$(D^n, S^{n-1}) \xrightarrow{c} (D^n \vee D^n, S^{n-1} \vee S^{n-1})$$




Then $[f]+[g]$ in $\pi_n(X, A)$ where

$f, g : (D^n, S^{n-1}) \longrightarrow (X, A)$, is

$[(f,g)\circ c]$ where

$$(f,g) : (D^n \vee D^n, S^{n-1} \vee S^{n-1}) \longrightarrow (X, A)$$

has the obvious meaning.

We have maps $(D^n \vee D^n, S^{n-1} \vee S^{n-1}) \xrightarrow{q_i} (D^n, S^{n-1})$

$q_1 = (\mathrm{id}, \mathrm{const})$, $q_2 = (\mathrm{const}, \mathrm{id})$

and $j_i : (D^n, S^{n-1}) \longrightarrow (D^n \vee D^n, S^{n-1} \vee S^{n-1})$

given by the 2 inclusions, and we have

$q_i \circ c \simeq \mathrm{id}_{(D^n, S^{n-1})}$ as maps of pairs

and $q_i \circ j_k = \delta_{ik} \cdot \mathrm{id}_{(D^n, S^{n-1})}$ where $0 \cdot \mathrm{id} = \mathrm{const}$.

we have to show that

$$H_n((f,g)\circ c)(\iota_n) = H_n(f)(\iota_n) + H_n(g)(\iota_n)$$

Look at

$$\begin{tikzcd}
H_n((D^n, S^{n-1})) \arrow[r, "H_n(c)"] & H_n(D^n \vee D^n, S^{n-1} \vee S^{n-1}) \arrow[r, "H_n((f,g))"] \arrow[d, shift left, "\begin{pmatrix} H_n(q_1) \\ H_n(q_2) \end{pmatrix}", "\cong"'] & H_n(X, A) \\
& H_n(D^n, S^{n-1}) \oplus H_n(D^n, S^{n-1}) \arrow[u, shift left, "(H_n(j_1), H_n(j_2))", "\cong"'] &
\end{tikzcd}$$

Then $\rightarrow\downarrow$ is the diagonal, and

$\uparrow\rightarrow$ maps $(x, y)$ to $H_n(f)(x) + H_n(g)(y)$

since $(f,g)\circ j_1 = f$, $(f,g)\circ j_2 = g$. □

<u>Remark</u>: If fundamental groups are involved the Hurewicz theorem has to be modified. For example, for path connected $X$ there is no isomorphism

$$\pi_1(X, x_0) \longrightarrow \underset{\tilde{H}_1(X)}{\overset{\cong}{H_1(X, x_0)}}$$

unless $\pi_1(X, x_0)$ is abelian;