Then our arguments show

6.4 Proposition: Every Ω-spectrum $\mathbb{K} = (K_n, K_n \to \Omega K_{n+1})$ defines a reduced cohomology theory

$$h^n := [-, K_n]_0$$

on the category of pointed CW-complexes. □

Remark: The groups $h^n(S^0)$ are called the coefficient groups for the cohomology theory h.

Our next step is a uniqueness theorem for cohomology (and homology) theories when restricted to CW-complexes. The usual formulation is for unreduced unpointed theories. There are ways to pass from reduced pointed to unreduced (and unpointed) and from there to reduced unpointed theories as follows.

Let $\tilde{h}^*$ be reduced, pointed.

Define for a CW-pair

$$h^n(X, A) := \tilde{h}^n(X/A) \quad \text{with } [A] \text{ as basepoint}$$

and where $h^n(X) := \tilde{h}^n(X/\emptyset) = \tilde{h}^n(X_+)$ with $X_+ = X \sqcup$ base point. This is an unpointed theory and $h^n(pt) = \tilde{h}^n(pt_+) = \tilde{h}^n(S^0)$.

From an unreduced unpointed theory one obtains a reduced unpointed theory by setting

$$\tilde{h}^n(X) := \operatorname{coker}(h^n(pt) \longrightarrow h^n(X)), \text{ where}$$

$X \to pt$ is the unique constant map.

There is an obvious analogue for homology theories

6.5 Theorem. Let h_* (h^*) be an unreduced homology (cohomology) theory on the category of CW-complexes and continuous maps with $h_n(pt) = 0$ for $n \neq 0$ ($h^n(pt) = 0$ for $n \neq 0$). Then h_n (h^n) and $H_n(-; h_0(pt))$ ($H^n(-; h^0(pt))$) are naturally isomorphic.

Proof. The key observation is that h_n (or h^n) can be computed from the cellular complex

$$\begin{tikzcd}
e^h_n := h_n(X^{(n)}, X^{(n-1)}) \arrow[dd] \arrow[dr, "\partial"] & \cong h_n(X^{(n)}/X^{(n-1)}, *) \\
& h_{n-1}(X^{(n-1)}) \arrow[dl] \quad \cong \tilde{h}_n(\vee S^n) \\
e^h_{n-1} := h_{n-1}(X^{(n-1)}, X^{(n-2)}) &
\end{tikzcd}$$

using the same arguments as for singular homology (or cohomology).

Since we have a suspension isomorphism

$$\tilde{h}_{n+1}(SX) = \tilde{h}_{n+1}(CX/X) \longrightarrow \tilde{h}_n(X)$$

since the terms to the left and right are reduced homology groups of a point, we have by the wedge axiom that the cellular complexes for

$H_*(-; h_0(pt))$ and h_* are degree wise isomorphic, so that one only has to compare their corresponding boundary maps

For $H_*(-;G)$ we know how these are related to degrees of maps $S^n \to S^n$. Thus it suffices to prove:

<u>6.5.1 Lemma</u>: Let $f: S^n \to S^n$ be a map of degree k. Then $h_n(f): h_n(S^n) \longrightarrow h_n(S^n)$ is multiplication by k.

The proof of this Lemma is completely analogous to the proof (see 5.10) that the Hurewicz map is a homomorphism, so we omit it here. $\square$

Thus we obtain isomorphisms $h_n(X) \to H_n(X;G)$, $G := h_0(pt)$, for every CW-cx. X. To see that this is natural first replace

$$f: X \to Y$$

by a cellular map. Then f induces a chain map of the cellular complexes for h_* and $H_*(-;G)$. Again their effect is determined by degrees of maps between spheres in h-theory

$$h_n(X^{(n)}/X^{(n-1)}) \longrightarrow h_n(Y^{(n)}/Y^{(n-1)})$$

and $H_*(-;G)$-theory. So they are the same for both theories.

<u>Remark</u>. This argument needs some additional consideration for cohomology theories.

A cochain on an infinite generated free abelian group might have non-zero values on infinitely many generators as well as $\tilde{h}^n(\bigvee_{\alpha\in A} S^n_\alpha) \cong \prod_{\alpha\in A} (\tilde{h}^n(S^n_\alpha)) \hat{=} \prod_{\alpha\in A} h^0(pt)$

will have elements where every component is non-zero (if $h^0(pt) \neq 0$).

Fortunately, for the (co-)boundary maps of the cellular complex this causes no problem, since the attaching maps involve only finitely many cells.

For $H^n(-;G)$ we know what happens since the cochain complex is just $\mathrm{Hom}(-;G)$ of the cellular chain complex.

For h^* we look at

$$\begin{tikzcd}
h^n(X^{(n)}, X^{(n-1)}) \cong \prod\limits_{n\text{-cells}} \tilde{h}^n(S^n) \arrow[dr] & & \\
& h^n(X^{(n)}) \arrow[r, "h^n(\mathrm{attach})"] \arrow[dl, "\delta"'] & h^n(S^n) \hat{=} G \arrow[d, "\delta"] \\
h^{n+1}(X^{(n+1)}, X^{(n)}) \arrow[rr, "h^{n+1}(\mathrm{char})"] & & h^{n+1}(D^{n+1}, S^n) \cong G
\end{tikzcd}$$

$h^{n+1}(X^{(n+1)}, X^{(n)}) \cong \prod\limits_{(n+1)\text{-cells}} \tilde{h}^{n+1}(S^{n+1})$

Using the fact that the attaching map lies in the union of finitely many cells we see that each component of $\delta(\varphi)$ depends only on finitely many components of $\varphi \in \prod\limits_{n\text{-cells}} \tilde{h}^n(S^n)$ and if these components are $g_1, \dots, g_r$ and the degree of the attaching map for the n-cell corresponding to g_i is d_i then the component corresponding to the attached $(n+1)$-cell is $\sum_1^r d_i g_i$. So the calculation is exactly as for $H^*(-;G)$, again using the fact that for a degree d map $f: S^n \to S^n$ we have $\tilde{h}^n(f)(g) = d \cdot g$, $g \in G \cong \tilde{h}^n(S^n)$.

This argument also applies for the proof that the isomorphism $h^n(X,A) \longrightarrow H^n(X,A;G)$ is natural. $\square$

6.6 Corollary: The cohomology theories associated to $\tilde{h}^n(X) = [X; K(G,n)]$ and $H^n(X;G)$ are naturally isomorphic on the category of pointed CW-complexes.

Proof: $\tilde{h}^n(S^0) \cong \pi_0(K(G,n)) \cong \begin{cases} G & n=0 \\ 0 & n\neq 0. \end{cases}$

That the isomorphism is given by some element u of $H^n(K(G,n);G)$ follows from naturality.

Take $u \in H^n(K(G,n);G)$ as the image of $[id] \in [K(G,n), K(G,n)]$. Then consider for $f: X \longrightarrow K(G,n)$ the diagram

$$\begin{tikzcd}
{[f]} & {[X, K(G,n)]} \arrow[r, "\text{iso}"] & H^n(X;G) & f^*(u) \\
{[id]} \arrow[u, mapsto] & {[K(G,n), K(G,n)]} \arrow[u, "f^*"] \arrow[r, "\text{iso}"] & H^n(K(G,n);G) \arrow[u, "f^*"'] & u \arrow[u, mapsto] \\
& {[id]} \arrow[r, mapsto] & u &
\end{tikzcd}$$

which by naturality commutes, and follow $[id]$ on its two ways to $H^n(X;G)$ $\square$

As a final topic: some remarks on homology and spectra.

Here it is useful to change the axioms for a (co-)homology theory into an equivalent system. Again we consider only basepointed reduced theories.

Then a (red) cohomology theory is given by a sequence of contravariant functors $h^n : CW_* \longrightarrow$ Ab. Groups

together with a natural isom. (red. susp.)

$$\sigma^n(X): h^n(X) \longrightarrow h^{n+1}(\Sigma X) \quad \text{s.t.}$$

(i) homotopy invariance

(ii) For each pair (X,A) $\quad h^n(X/A) \to h^n(X) \to h^n(A)$ is exact

(iii) Wedge Axiom

For homology simply require covariance

(i) as above

(ii) $h_n(A) \to h_n(X) \to h_n(X/A)$ exact

(iii) $\bigoplus_{j\in J} h_n(X_j) \xrightarrow[\cong]{} h_n(\bigvee_{j\in J} X_j)$ is

an isom., where $\to$ is the natural map induced by the inclusions of the X_j in the wedge

This suffices to give a reduced (co-) homology theory as defined before (which induced the natural isom.

$h^n(X) \to h^{n+1}(\Sigma X)$).

What is missing is now $h^n(A) \to h^{n+1}(X/A)$ making the long sequence exact. Here we look at the cofibration sequence

$$A \to X \to X/A \xrightarrow{q} \Sigma A \to \Sigma X \to \cdots$$

and apply h^n to it to get our map

$$h^n(A) \xrightarrow[\approx]{\sigma^n} h^{n+1}(\Sigma A) \xrightarrow{h^{n+1}(q)} h^{n+1}(X/A).$$

At each step we have a sequence of the type

$$B \to Y \to Y/B$$

which is exact by (ii).

An Ω-spectrum was a sequence of spaces K_n, $n \in \mathbb{Z}$, together with (weak) homotopy equivalences

$$K_n \longrightarrow \Omega K_{n+1}.$$

We know that there is a natural bijection $[\Sigma K_n ; K_{n+1}] \longrightarrow [K_n, \Omega K_{n+1}]$. This leads us to the definition of a (naive) spectrum:

__6.7 Definition:__ A spectrum is a sequence of (pointed) CW-complexes K_n, $n \in \mathbb{Z}$, together with maps $\Sigma K_n \longrightarrow K_{n+1}$.

A map between spectra is the obvious thing i.e. maps $f_n : K_n \to K_n'$ s.t.

$$\begin{tikzcd} \Sigma K_n \arrow[r] \arrow[d, "\Sigma f_n"'] & K_{n+1} \arrow[d, "f_{n+1}"] \\ \Sigma K_n' \arrow[r] & K_{n+1}' \end{tikzcd}$$

commutes.

__Examples:__ (a) X a space

$$K_n = \Sigma^n X \quad , \quad n \geq 0$$
$$K_n = pt \qquad n < 0$$

$\Sigma K_n = K_{n+1}$

this is called the suspension spectrum.

(b) If $\{K_n\} = \mathbb{K}$ is a spectrum we can form

$\mathbb{K} \wedge X$ with $(\mathbb{K} \wedge X)_n = K_n \wedge X$ with the obvious $\Sigma K_n \wedge X \longrightarrow K_{n+1} \wedge X$ as structure maps.

Now consider for a spectrum $\mathbb{K}$ the sequence of maps

$$\pi_i(K_n) \xrightarrow{\text{susp.}} \pi_{i+1}(\Sigma K_n) \longrightarrow \pi_{i+1}(K_{n+1}) \rightarrow$$

i.e. we have associated maps

$$\pi_i(K_n) \rightarrow \pi_{i+1}(K_{n+1})$$

and can form the direct limit (colimit) of this sequence. This we define to be the $(i-n)$-th homotopy group of $\mathbb{K}$

6.7.1. Definition. $\mathbb{K} = \{K_n\}$ a spectrum.

Then $\pi_i(\mathbb{K}) = \operatorname{colim}(\rightarrow \pi_{n+i}(K_n) \longrightarrow \pi_{n+i+1}(K_{n+1}) \rightarrow$

<u>6.8 Theorem:</u> Let $\mathbb{K}$ be a spectrum. Then the functors

$$X \longmapsto \pi_n(X \wedge \mathbb{K}) \qquad n \in \mathbb{Z}$$

define a (reduced pointed) homology theory.

The suspension isomorphism is practically built into the definition of the homology theory.

$\pi_n(X \wedge \mathbb{K})$ is the colimit of

$$\begin{tikzcd}
\pi_{n+i}(X \wedge K_i) \arrow[r] \arrow[ddr] & \pi_{n+i+1}(\Sigma(X \wedge K_i)) \arrow[d, equal] \\
& \pi_{n+i+1}(X \wedge \Sigma K_i) \arrow[d] \\
& \pi_{n+1+i}(X \wedge K_{i+1}) \arrow[d] \\
& \pi_{n+1+i+1}(\Sigma(X \wedge K_{i+1})) \arrow[d, equal] \\
& \pi_{n+1+i+1}(\Sigma X \wedge K_{i+1})
\end{tikzcd}$$

and $\pi_{n+1}((\Sigma X) \wedge K)$ is the colimit looking at the composition of the vertical arrows.

<u>Remark:</u> Any cohom. theory is represented by an Ω-spectrum (Brown-representation)

Any homology theory satisfying the compact support axiom, i.e.

$$h_n(X) \xleftarrow{\cong} \operatorname{colim}_f (h_n(X_f)) \quad \text{where } X_f \text{ runs}$$

through finite subcomplexes, is represented by a spectrum.