

Correction to Lectures 5 and 6:

Proposition 5.11

Thanks to Peter Patet's prodding we discovered a real blooper in the formulation of Proposition 5.11 and again thanks to some input of Peter we have now a short and easy proof of this proposition.

First the blooper: we want S_k to act on $F_k(M)$, so that for $\tau, \sigma \in S_k$ we have $(\tau \circ \sigma)(x) = \tau(\sigma(x))$, $x \in F_k(M)$. Thinking of x as embedding

$$x: \{1, \dots, k\} \rightarrow M$$

we defined σx in Proposition 5.11 by $x \circ \sigma$ so that

$$\tau \circ \sigma(x) = x \circ \tau \circ \sigma = \sigma(\tau(x)).$$

Thus: if we prove the proposition for transpositions ($\tau \tau^{-1}$) only, we will obtain for $\sigma = \tau_p \circ \dots \circ \tau_1$, where each τ_i is a transposition

$$\sigma \circ \alpha'_{rs} = \tau_1 \circ \tau_2 \circ \dots \circ \tau_p \circ \alpha'_{rs}$$

$$\cong \alpha'_{\tau_1 \circ \dots \circ \tau_p r, \tau_1 \circ \dots \circ \tau_p s} = \alpha'_{\sigma^{-1} r, \sigma^{-1} s}$$

Thus to keep a nice equation (and since we like to write maps from the left) we define

$$\sigma x = \{1, \dots, k\} \rightarrow M \quad \text{by}$$

$$x \circ \sigma^{-1}$$

Since $\sigma = \sigma^{-1}$ for transpositions, we have proved

5.11 with the correction (not the the original one of 5.11).

Second: Peter suggested slightly different representatives

α'_{rs} for α_{rs} ; one that allows to do away with distinguishing the cases $r > s, s > r$.

Thus: fix $r \neq s$ with $1 \leq r, s \leq k$ and set

$$\alpha'_{rs}(\xi) = (q_1, \dots, q_{r-1}, q_s + \xi, q_{r+1}, \dots, q_k), \quad \xi \in S^n$$

Main advantage: works properly for both cases

and α'_{rs} new is homotopic to α_{rs} old, so the α_{rs} do not change.

Now Proposition 5.11 reads as follows:

Let $\sigma \in S_k$ act on $F_k(\mathbb{R}^{n+1})$ by

$$\sigma(x_1, \dots, x_k) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)})$$

$$\text{then } \pi_n(\sigma)(\alpha_{rs}) = \alpha_{\sigma(r)\sigma(s)}$$

Proof: For any map $h: X \rightarrow F_k(M) \subset \underbrace{M \times \dots \times M}_k$ we denote the i -th component by h_i .

$$\text{Then } (\sigma \circ \alpha'_{rs}(\xi))_i = \begin{cases} q_{\sigma^{-1}(i)} & , \sigma(r) \neq i \\ q_s + \xi & , \sigma(r) = i \end{cases}$$

we also note that

$$(\sigma \circ \alpha'_{rs}(\xi))_{\sigma(s)} = q_s$$

On the other hand

G.A.3

$$\left(\alpha'_{\sigma(r)\sigma(s)} \left(\begin{matrix} \xi \\ \xi \end{matrix} \right) \right)_i = \begin{cases} q_i & , i \neq \sigma(r) \\ q_{\sigma(s)} + \xi & , i = \sigma(r) \end{cases}$$

we also note that

$$\left(\alpha'_{\sigma(r)\sigma(s)} \left(\begin{matrix} \xi \\ \xi \end{matrix} \right) \right)_{\sigma(s)} = q_{\sigma(s)}$$

Denote $\alpha'_{\sigma(r)\sigma(s)}$ by f_0 .

Then f_0 is homotopic to f_1 where

$$f_t i = f_0 i + (1-t)(q_s - q_{\sigma(s)})$$

(i.e. we add in each component the same element of \mathbb{R}^{n+1}). Thus this is a homotopy of maps into $F_k(\mathbb{R}^{n+1})$.

This simple homotopy is due to Peter and makes the rest of the proof very simple.

Notice:

$$f_{1t} \left(\begin{matrix} \xi \\ \xi \end{matrix} \right) = \begin{cases} q_i + q_s - q_{\sigma(s)} =: \gamma_i & i \neq \sigma(r) \\ q_s - \xi & i = \sigma(r) \end{cases}$$

$$\text{and } f_{1t} \left(\begin{matrix} \xi \\ \xi \end{matrix} \right)_{\sigma(s)} = q_s$$

Notice that γ_i , $i \neq \sigma(r), \sigma(s)$, is in the complement of $q_s + B^{n+1}$, where B^{n+1} is the unit ball in \mathbb{R}^{n+1} .

The same holds for $q_{\sigma^{-1}(i)} = \sigma \circ \alpha'_{rs} \left(\begin{matrix} \xi \\ \xi \end{matrix} \right)_i$ if $i \neq \sigma(r), \sigma(s)$.

