Configuration spaces

Problem Set 3
WS 2013/14

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Exercise 1
Show that $\mathbb{R}^n - \{x_1, \dots, x_k\}$ is homotopy equivalent to $\vee_{i=1}^k S^1$.

Exercise 2
Consider the fiber bundle

$$\mathbb{R}^2 - \{0,1\} \to F_2(\mathbb{R}^2 - \{0\}) \xrightarrow{p} F_1(\mathbb{R}^2 - \{0\}) \cong \mathbb{R}^2 - \{0\}$$

with $p(x_1, x_2) = x_1$. Show that this fiber bundle is trivial, i.e. there is a homeomorphism

$$\psi : F_2(\mathbb{R}^2 - \{0\}) \to (\mathbb{R}^2 - \{0\}) \times (\mathbb{R}^2 - \{0,1\})$$

with $p = \pi_{\mathbb{R}^2 - \{0\}} \circ \psi$.

Exercise 3
Let $e_1 \in \mathbb{R}^3$ be the first unit vector. Show that the map $p\colon SO(3) \to S^2, A \mapsto Ae_1$ is a fiber bundle. Furthermore, show that there is no map $s\colon S^2 \to SO(3)$ with $p \circ s = Id_{S^2}$.
Hint: You may use that S^2 has no nowhere-vanishing tangent vector field.

Exercise 4
Consider the fiber bundle

$$\mathbb{R}^3 - \{0, e_1\} \to F_2(\mathbb{R}^3 - \{0\}) \xrightarrow{p} F_1(\mathbb{R}^3 - \{0\}) \cong \mathbb{R}^3 - \{0\}$$

Restricting p to $S^2 \subset \mathbb{R}^3$, we obtain the fiber bundle

$$S^2 - \{e_1\} \to F_2(S^2) \xrightarrow{p} F_1(S^2) \cong S^2$$

If this bundle were trivial, i.e. if there were a homeomorphism

$$\psi\colon F_2(S^2) \to S^2 \times (S^2 - \{e_1\})$$

such that $p \circ s = Id_S^2$, ψ would restrict to homeomorphisms of the fibers $\psi_x\colon p^{-1}(x) = \{y \in S^2 \mid y \neq x\} \to S^2 - \{e_1\}$ for all $x \in S^2$. Show that there is no such ψ such that each ψ_x is induced by a map $A \in SO(3)$, i.e. such that for each $x \in S^2$, there is $A(x) \in SO(3)$ such that $\psi_x(y) = A(x)y$ for all y.