Lecture 9

Supplement to the proof of 8.2 b

Recall the sequence of fibre bundles

$$\begin{tikzcd}
F_k(\mathbb{R}^{n+1}) \arrow[d, "p_0"] \arrow[r, hook, "\simeq"] & F_{k-1}(\mathbb{R}^{n+1}_1) \arrow[d, shift right, "p_1"'] \arrow[l, hook] & \cdots \arrow[l, hook] & F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}) \arrow[l, hook] \arrow[d, shift right, "p_{r-1}"'] & \cdots \arrow[l, hook] & F_1(\mathbb{R}^{n+1}_{k-1}) \arrow[l, hook] \arrow[d, shift right, "p_{k-1}=id"'] \\
\mathbb{R}^{n+1} & \mathbb{R}^{n+1}_1 \arrow[u, shift right, "s_1"'] & & \mathbb{R}^{n+1}_{r-1} \arrow[u, shift right, "s_{r-1}"'] & & \mathbb{R}^{n+1}_{k-1} \arrow[u, shift right, "s_{k-1}=id"']
\end{tikzcd}$$

$\pi_*(s_{r-1})\left(\pi_*(\mathbb{R}^{n+1}_{r-1})\right)$ is a sub Liealgebra of

$$\pi_*\left(F_{k-r+1}(\mathbb{R}^{n+1}_{r-1})\right) \subset \pi_*\left(F_k(\mathbb{R}^{n+1})\right)$$

which we denoted by $L_{r,r-1}$. In degree n $L_{r,r-1}$ has the basis $\alpha_{r1}, \dots, \alpha_{rr-1}$. In particular $[\alpha_{rs}, \alpha_{rt}] \in L_{r,r-1}$

Additively we had

$$\pi_*(F_k(\mathbb{R}^{n+1})) \cong \bigoplus_{r=2}^{k} L_{r,r-1}$$

(*) Therefore, an equation $0 = x_1 + x_2 + \dots + x_{k-1}$ with $x_i \in L_{i+1,i}$ implies $x_i = 0$ for all i.

$$L_{r,r-1} \cong \pi_*(\mathbb{R}^{n+1}_{r-1}) \cong \pi_*(S^n_1 \vee \dots \vee S^n_{r-1})$$

$$\pi_n(S^n_1 \vee \dots \vee S^n_{r-1}) \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{r-1}$$

with basis $\iota_1, \dots, \iota_{r-1}$ where

ι_i is represented by the inclusion $S^n_i \hookrightarrow S^n_1 \vee \dots \vee S^n_{r-1} =: X$

(recall: each S^n_i is a copy of S^n)

We are interested in

$\pi_{2n-1}(X)$. We have

$$\begin{tikzcd} S^n_i \arrow[r, hook, "j_i"] \arrow[dr, "q_i \circ j_i = id"'] & X \arrow[d, "q_i"] \\ & S^n_i \end{tikzcd}$$

where q_i is the identity on S^n_i and maps all other summands to the basepoint.

So we see: for each i

$\pi_{2n-1}(X) = \ker(q_{i*}) \oplus \pi_{2n-1}(S^n_i)$, the map induced in π_{2n-1} by q_i.

Clearly, $\pi_{2n-1}(S^n_t) \subset \ker(q_{i*})$ if $t \neq i$. Thus we get

$$\pi_{2n-1}(X) = \bigcap_{i=1}^{r-1} \ker(q_{i*}) \oplus \bigoplus_{i=1}^{r-1} \pi_{2n-1}(S^n_i).$$

Now consider

$$[\iota_i, \iota_t] \in \pi_{2n-1}(S^n_i \vee S^n_t)$$

$1 \le i < t \le r-1$. Using

$$\begin{tikzcd} S^n_i \vee S^n_t \arrow[r, hook, "j_{i,t}"] \arrow[dr, "id"'] & X \arrow[d, "q_{i,t}"] \\ & S^n_i \vee S^n_t \end{tikzcd}$$

and 7.2 + Exercise 1 on problem set 6 we see that

$[\iota_i, \iota_t] \in \pi_{2n-1}(X)$ is an element of infinite order which is not in the kernel of $q_{i,t*}$.

If $1 \le i' < t' \le r-1$ and $(i',t') \neq (i,t)$ then $[\iota_{i'}, \iota_{t'}] \in \ker q_{i,t*}$ by naturality of $[\ ,\ ]$.

Also, every $[\iota_i, \iota_t] \in \ker q_{s*}$ for every $1 \le s \le r-1$.

(**) Therefore, if we have an equation in $\pi_{2n-1}(S_1^n \vee \cdots \vee S_{r-1}^n)$ of the form

$$0 = x_1 + \cdots + x_{r-1} + \sum_{1 \le i < t \le r-1} c_{it} [\iota_i, \iota_t]$$

with $x_i \in \pi_{2n-1}(S_i^n)$ then all $x_i = 0$ and all $c_{it} = 0$.

(Proof: Apply q_{i*}, $i = 1, \dots, r-1$ and $q_{i,t*}$, $1 \le i < t \le r-1$ to the right hand side)

Recall now the proof of 8.2.(b). first part, where we prove that $n = 3, 7$ if p_{rk} is fibre homotopy trivial.

We get an equation

$$0 = [\phi(\alpha_{21}), \alpha_{r+11}] = [\alpha_{21} + \beta_{r+1} + \sum_{s=r+2}^{k} \beta_s \ ,\ \alpha_{r+11}]$$

where $\beta_s \in L_{s,s-1}$ in degree n. We used Y.-B. relations to show $[\beta_s, \alpha_{r+11}] \in L_{s,s-1}$, $s = r+1, r+2, \dots, k$ and $[\alpha_{21}, \alpha_{r+11}] \in L_{r+1,r}$

Using (*) on page 9.1. we see that

$[\beta_s, \alpha_{r+1\,1}] = 0$ for $s = r+2, \dots, k$ and

$[\alpha_{21} + \beta_{r+1}, \alpha_{r+1\,1}] = 0$.

Now we wrote for $\beta_{r+1} = \sum_{i=1}^{r} c_i \alpha_{r+1\,i}$ since

$\beta_{r+1} \in (L_{r+1,r})_n$ which has $\alpha_{r+1\,1}, \dots, \alpha_{r+1\,r}$ as a basis.

We used Yang Baxter to show that

$[\alpha_{21}, \alpha_{r+1\,1}] = (-1)^n [\alpha_{r+1\,2}, \alpha_{r+1\,1}]$ and therefore

obtain

$$c_1 [\alpha_{r+1\,1}, \alpha_{r+1\,1}] + ((-1)^n + c_2)[\alpha_{r+1\,2}, \alpha_{r+1\,1}]$$

$$+ \sum_{i=3}^{r} c_i [\alpha_{r+1\,i}, \alpha_{r+1\,1}] = 0$$

Translated into

$\pi_{2n-1}(\mathbb{R}^{n+1}_r) \cong \pi_{2n-1}(S^n_1 \vee \dots \vee S^n_r)$ we get

the equation

$$0 = c_1 [\iota_1, \iota_1] + ((-1)^n + c_2)[\alpha_{r+1\,2}, \alpha_{r+1\,1}] + \sum_{i=3}^{r} c_i [\alpha_{r+1\,i}, \alpha_{r+1\,1}]$$

Using (**) we get

$c_1 [\iota_1, \iota_1] = 0$, $c_2 = (-1)^{n+1}$, $c_i = 0$ for $i \geq 3$.

Then we do the same calculation for

$0 = [\phi(\alpha_{21}), \alpha_{r+1\,2}]$ to obtain $c_1 = (-1)^{n+1}$, $c_2[\iota_2, \iota_2] = 0$.

Thus $[\iota_1, \iota_1] = 0$ which implies that S^n is an H-space which implies $n = 1, 3, 7$. Since $n \geq 2$, we get $n = 3, 7$. The rest of the proof (i.e. $r = 2$) should be clear. □