

Supplement to the proof of 8.2 b

Recall the sequence of fibre bundles

$$\begin{array}{ccccccc}
 F_k(\mathbb{R}^{n+1}) & \xleftrightarrow{\cong} & F_{k-1}(\mathbb{R}_1^{n+1}) & \xleftrightarrow{\cong} & \dots & \xleftrightarrow{\cong} & F_{k-r+1}(\mathbb{R}_{r-1}^{n+1}) & \xleftrightarrow{\cong} & \dots & \xleftrightarrow{\cong} & F_1(\mathbb{R}_{k-1}^{n+1}) \\
 \downarrow p_0 & & \downarrow p_1 & \uparrow s_1 & & & \downarrow p_{r-1} & \uparrow s_{r-1} & & & \downarrow p_{k-1} & \uparrow s_{k-1} \\
 \mathbb{R}^{n+1} & & \mathbb{R}_1^{n+1} & & & & \mathbb{R}_{r-1}^{n+1} & & & & \mathbb{R}_{k-1}^{n+1}
 \end{array}$$

$\pi_*(S_{r-1})(\pi_*(\mathbb{R}_{r-1}^{n+1}))$  is a sub Lie algebra of

$$\pi_*(F_{k-r+1}(\mathbb{R}_{r-1}^{n+1})) \subset \pi_*(F_k(\mathbb{R}^{n+1}))$$

which we denoted by  $L_{r,r-1}$ . In degree  $n$   $L_{r,r-1}$  has

the basis  $\alpha_{r-1}, \dots, \alpha_{r-2}$ . In particular  $[\alpha_{r-1}, \alpha_{r-2}] \in L_{r,r-1}$

Additively we had

$$\pi_*(F_k(\mathbb{R}^{n+1})) \cong \bigoplus_{r=2}^k L_{r,r-1}$$

(\*) Therefore, an equation  $0 = x_1 + x_2 + \dots + x_{r-2}$

with  $x_i \in L_{i+1,i}$  implies  $x_i = 0$  for all  $i$ .

$$L_{r,r-1} \cong \pi_*(\mathbb{R}_{r-1}^{n+1}) \cong \pi_*(S_1^n \vee \dots \vee S_{r-1}^n)$$

$$\pi_n(S_1^n \vee \dots \vee S_{r-1}^n) \cong \underbrace{\mathbb{Z} \oplus \dots \oplus \mathbb{Z}}_{r-1}$$

with basis  $L_1, \dots, L_{r-1}$  where

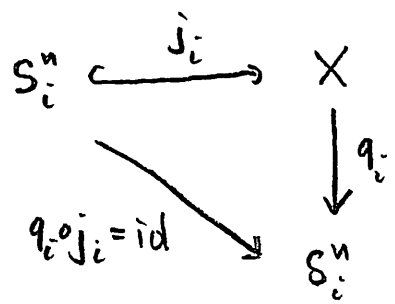
$L_i$  is represented by the inclusion  $S_i^n \hookrightarrow S_1^n \vee \dots \vee S_{r-1}^n =: X$

(recall: each  $S_i^n$  is a copy of  $S^n$ )

We are interested in

$$\pi_{2n-1}(X).$$

We have



where  $q_i$  is the identity on  $S_i^n$  and maps all other summands to the basepoint.

So we see: for each  $i$

$$\pi_{2n-1}(X) = \ker(q_{i*}) \oplus \pi_{2n-1}(S_i^n), \text{ the map induced in } \pi_{2n-1} \text{ by } q_i.$$

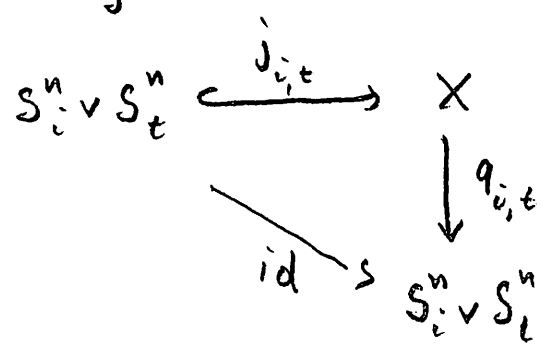
Clearly,  $\pi_{2n-1}(S_t^n) \subset \ker(q_{i*})$  if  $t \neq i$ . Thus we get

$$\pi_{2n-1}(X) = \bigcap_{i=1}^{r-1} \ker(q_{i*}) \oplus \bigoplus_{i=1}^{r-1} \pi_{2n-1}(S_i^n).$$

Now consider

$$[c_i, c_t] \in \pi_{2n-1}(S_i^n \vee S_t^n)$$

$1 \leq i < t \leq r-1$ . Using



and 7.2 + Exercise 1 on problem set 6 we see that

$[c_i, c_t] \in \pi_{2n-1}(X)$  is an element of infinite order

which is not in the kernel of  $q_{i,t} \neq$

If  $1 \leq i' \leq t' \leq r-1$  and  $(i', t') \neq (i, t)$  then

$[c_{i'}, c_{t'}] \in \ker q_{i,t} \neq$  by naturality of  $[ , ]$ .

Also, every  $[c_i, c_t] \in \ker q_{s \neq}$  for every

$$1 \leq s \leq r-1.$$

(\*\*) Therefore, if we have an equation in  $\pi_{2n-1}(S_1^n \vee \dots \vee S_{r-1}^n)$  of the form

$$0 = x_1 + \dots + x_{r-1} + \sum_{1 \leq i < t \leq r-1} c_{it} [c_i, c_t]$$

with  $x_i \in \pi_{2n-1}(S_i^n)$  then

all  $x_i = 0$  and all  $c_{it} = 0$ .

(Proof: Apply  $q_{i \neq}$ ,  $i=1, \dots, r-1$  and  $q_{i,t \neq}$ ,  $1 \leq i < t \leq r-1$  to the right hand side)

Recall now the proof of 8.2.(b). first part, where we prove that  $n=3, 7$  if  $p_{r \neq}$  is fibre homotopy trivial.

We get an equation

$$0 = [\phi(\alpha_{2+1}), \alpha_{r+1+1}] = [\alpha_{2+1} + \beta_{r+1} + \sum_{s=r+2}^k \beta_s, \alpha_{r+1+1}]$$

where  $\beta_s \in L_{s, s-2}$  in degree  $n$ . We used  $\gamma = \beta$  relations

to show  $[\beta_s, \alpha_{r+1+1}] \in L_{s, s-1}$ ,  $s = r+1, r+2, \dots, k$

and  $[\alpha_{2+1}, \alpha_{r+1+1}] \in L_{r+1, r}$

Using (\*) on page 9.1. we see that

$$[\beta_s, \alpha_{r+1}] = 0 \text{ for } s = r+2, \dots, k \quad \text{and}$$

$$[\alpha_{2r} + \beta_{r+1}, \alpha_{r+1}] = 0.$$

Now we write for  $\beta_{r+1} = \sum_{i=1}^r c_i \alpha_{r+1+i}$  since

$\beta_{r+1} \in (L_{r+1,r})_n$  which has  $\alpha_{r+1}, \dots, \alpha_{r+1+r}$  as a basis.

We used Yang Baxter to show that

$$[\alpha_{2r}, \alpha_{r+1}] = (-1)^n [\alpha_{r+2}, \alpha_{r+1}] \quad \text{and therefore}$$

obtain

$$c_1 [\alpha_{r+2}, \alpha_{r+1}] + ((-1)^n + c_2) [\alpha_{r+2}, \alpha_{r+1}] + \sum_{i=3}^r c_i [\alpha_{r+1+i}, \alpha_{r+1}] = 0$$

Translated into

$$\pi_{2n-1}(\mathbb{R}_r^{n+1}) \cong \pi_{2n-1}(S_1^n \vee \dots \vee S_r^n) \quad \text{we get}$$

the equation

$$0 = c_1 [e_1, e_2] + ((-1)^n + c_2) [\alpha_{r+2}, \alpha_{r+1}] + \sum_{i=3}^r c_i [\alpha_{r+1+i}, \alpha_{r+1}]$$

Using (\*\*\*) we get

$$c_1 [e_1, e_2] = 0, \quad c_2 = (-1)^{n+1}, \quad c_i = 0 \text{ for } i \geq 3.$$

Then we do the same calculation for

$$0 = [\phi(\alpha_{2r}), \alpha_{r+2}] \text{ to obtain } c_1 = (-1)^{n+1}, \quad c_2 [e_2, e_2] = 0.$$

Thus  $[e_2, e_2] = 0$  which implies that  $S^n$  is an H-space

which implies  $n = 1, 3, 7$ . Since  $n \geq 2$ , we get  $n = 3, 7$ . The rest of the proof ( $i = r+2$ ) should be clear.  $\square$