8. The homotopy Lie algebra of $\pi_*(F_k(\mathbb{R}^{n+1}))$, $n\geq 2$

Part V

We use the notation of the previous Lecture(s)

Fibration sequence and their sections

$$\begin{tikzcd} F_k(\mathbb{R}^{n+1}) \arrow[d, "p_0"] & F_{k-1}(\mathbb{R}^{n+1}_1) \arrow[l, hook'] \arrow[d, "p_1"'] & \cdots \arrow[l, hook'] & F_2(\mathbb{R}^{n+1}_{k-2}) \arrow[l, hook'] \arrow[d, "p_{k-2}"'] & F_1(\mathbb{R}^{n+1}_{k-1}) \arrow[l, hook'] \arrow[d, "id"] \\ \mathbb{R}^{n+1} & \mathbb{R}^{n+1}_1 \arrow[u, dashed, bend right, "s_1"'] & & \mathbb{R}^{n+1}_{k-2} \arrow[u, dashed, bend right, "s_{k-2}"'] & \mathbb{R}^{n+1}_{k-1} \arrow[u, dashed, bend left, "id"] \end{tikzcd}$$

Maps

$$\alpha'_{rs} : S^n \longrightarrow F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}) \longrightarrow F_k(\mathbb{R}^{n+1})$$

$$\xi \longmapsto (q_1, \ldots, q_{r-1}, q_s + \xi, q_{r+1}, \ldots, q_k)$$

$1 \leq s \neq r \leq k$

α_{rs} the htpy class of α'_{rs}. We know that

$\{\alpha_{rs} : 1 \leq s < r \leq k\}$ is a basis of the free abelian gp.

$\pi_n(F_k(\mathbb{R}^{n+1}))$

$\{\alpha_{rs}\ ,\ 1\leq s<r\}$, is the image under $\pi_n(s_{r-2})$

of the basis $\{\bar{\alpha}_{r,s} : 1\leq s<r\}$ of $\pi_n(\mathbb{R}^{n+1}_{r-1})$

where $\bar{\alpha}_{r,s}$ is represented by $\bar{\alpha}'_{r,s} : S^n \longrightarrow \mathbb{R}^{n+1}_{r-1}$
$\xi \longmapsto q_s + \xi$.

Since the Whitehead product is natural, $\pi_*(s_{r-2})(\pi_* \mathbb{R}^{n+1}_{r-1})$ is a sub-Lie algebra $L_{r,r-1}$ of $L_{k-r+1} := \pi_*(F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}))$

isomorphic to $\pi_*(\mathbb{R}^{n+1}_{r-1})$.

The inclusion $F_{k-r}(\mathbb{R}^{n+1}_{r}) \hookrightarrow F_{k-r+1}(\mathbb{R}^{n+1}_{r-1})$

maps its Lie algebra L_{k-r} injectively onto its image, also denoted by L_{k-r} and L_{k-r} is an ideal in L_{k-r+1}.

The question arises whether the direct sum decomposition $L_{k-r+1} = L_{k-r} \oplus L_{r,r-1}$ of <u>graded abelian groups</u> is a direct sum decomposition of Lie algebras.

The answer is, apart from the trivial cases $L_k = L_{k-1} \oplus \{0\}$ and $L_1 = L_1 \oplus \{0\}$, no

<u>8.1</u> For $2 \leq r \leq k-2$ the splitting $L_{k-r+1} = L_{k-r} \oplus L_{r,r+1}$ is not a direct sum decomposition of Lie algebras.

<u>Remark</u>: This does not imply that we cannot find a section $s'_{r-1} : \mathbb{R}^{n+1}_{r-1} \longrightarrow F_{k-r+1}(\mathbb{R}^{n+1}_{r-1})$ such that $L_{k-r+1} = L_{k-r} \oplus \pi_*(s'_{r-1})(\pi_*(\mathbb{R}^{n+1}_{r-1}))$ is a direct sum decomposition of Lie algebras. (see below)

Proof of 8.1: We are looking at

$$\begin{tikzcd} F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}) \arrow[d, "p_{r-1}"] & F_{k-r}(\mathbb{R}^{n+1}_{r}) \arrow[l, hook'] \\ \mathbb{R}^{n+1}_{r-1} \arrow[u, bend left, dashed, "s_{r-1}"] & \end{tikzcd}$$

By hypothesis we have $1 \le s < r < t \le k-1$. Then

$$\alpha_{rs} \in L_{r,r-1} \quad ; \quad \alpha_{tr}, \alpha_{ts} \in L_{k-t+1} \subseteq L_{k-r}$$

The Yang-Baxter relation gives us

$$[\alpha_{ts}, \alpha_{rt} + \alpha_{rs}] = 0 \quad \text{i.e.}$$

$$[\alpha_{ts}, \alpha_{rs}] = - [\alpha_{ts}, \alpha_{rt}] = (-1)^n [\alpha_{ts}, \alpha_{tr}]$$

If $L_{k-r+1} = L_{k-r} \oplus L_{r,r-1}$ were a direct sum decomposition of Lie algebras $[\alpha_{ts}, \alpha_{rs}] = 0$, i.e.

$= [\alpha_{ts}, \alpha_{tr}] = 0$ α_{ts}, α_{tr} are in the image of $\pi_*(s_{t-1})$ which is injective, i.e.

$$[\bar{\alpha}_{t,s}, \bar{\alpha}_{tr}] \in \pi_{2n-1}(\mathbb{R}^{n+1}_{t-1}) \quad \text{is } 0.$$

Now, $\mathbb{R}^{n+1}_{t-1} \xrightarrow[r_{t-1}]{\simeq} \underbrace{S^n \vee S^n \vee \cdots \vee S^n}_{t-1}$ summands.

Since basepoints can be ignored

$r_{t-1} \circ \bar{\alpha}'_{ti} : S^n \longrightarrow S^n \vee \cdots \vee S^n$ is inclusion into the i-th summand; $\pi_*(r_{t-1})$ is an isomorphism.

With the notation of Lecture 7, Proposition 7.2

$$\iota_i = \pi_n(r_{t-1})\,\alpha_{ti}. \qquad \text{Thus}$$

$$0 \neq [\iota_s, \iota_r] = \pi_{2n-1}(r_{t-1})([\alpha_{ts}, \alpha_{tr}])$$ and we get a contradiction. $\square$

The final section of Chapter II of Fadell-Husseini deal with the question, which of the fibrations

$$p_{r,k} : F_k(\mathbb{R}^{n+1}) \longrightarrow F_r(\mathbb{R}^{n+1})$$
$$(x_1, \dots, x_k) \longmapsto (x_1, \dots, x_r)$$

are trivial. They use a weaker concept of triviality than ~~demanding~~ that there is

a map $F_k(\mathbb{R}^{n+1}) \xrightarrow{q_{r,k}} p_{r,k}^{-1}(q_1, \dots, q_r)$

$$= F_{k-r}(\mathbb{R}^{n+1}_r)$$

such that $F_k(\mathbb{R}^{n+1}) \xrightarrow{(p_{r,k},\, q_{r,k})} F_r(\mathbb{R}^{n+1}) \times F_{k-r}(\mathbb{R}^{n+1}_r)$

is a homeomorphism, i.e. that there exists an isomorphisms of fibre bundles

$$\begin{tikzcd}
F_k(\mathbb{R}^{n+1}) \arrow[rr] \arrow[dr, "p_{r,k}"'] & & F_r(\mathbb{R}^{n+1}) \times F_{k-r}(\mathbb{R}^{n+1}) \arrow[dl, "\text{proj. on 1st factor}"] \\
& F_r(\mathbb{R}^{n+1}) &
\end{tikzcd}$$

Since we are dealing with htpy information one can use the concept of fibre homotopy equivalence for fibrations.

Given fibrations over X.

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ p \downarrow & & \downarrow p' \\ X & \xrightarrow{id} & X \end{array}$$

A fibre htpy equivalence between them is a map $f : E \to E'$ over X (i.e. s.t. the diagram commutes) s.t.

$\exists\ g : E' \longrightarrow E$ over X s.t. $g \circ f \simeq id_E$, $f \circ g \simeq id_{E'}$ through homotopies over X. If $F = p^{-1}(x_0)$ is the fibre of p over the base point one says that $\begin{array}{c} E \\ p\downarrow \\ X \end{array}$

is fibre homotopy trivial if p is fibre homotopy equivalent

to $\quad \begin{array}{ccc} X \times F & \xrightarrow{P_X} & X \\ (x, f) & \longmapsto & x \end{array}$.

8.2 (a) If $r = 2$ and $n = 3$ or 7 then the fibre bundle $F_k(\mathbb{R}^{n+1}) \xrightarrow{P_{r,k}} F_r(\mathbb{R}^{n+1})$ is trivial

(b) If $P_{r,k}$ is fibre homotopy trivial, and $2 \leq r < k$, then $r = 2$ and $n = 3$ or 7 (recall that we are dealing only with the case $n \geq 2$)

Remark: We know that $F_k(\mathbb{R}^{n+1}) \xrightarrow{P_0} F_1(\mathbb{R}^{n+1})$ is trivial for all $n \geq 0$.

Proof of 8.2.a: $\mathbb{R}^4$ and $\mathbb{R}^8$ are so called division algebras over $\mathbb{R}$, the quaternions and octonions

$\mathbb{R}^4$ with quaternion

The quaternions can be described as a subalgebra of $M_2(\mathbb{C})$, the 2×2-matrices over $\mathbb{C}$. We write $\mathbb{H}$ for the quaternions to honour their discoverer Hamilton.

Then

$$\mathbb{H} = \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} : u, v \in \mathbb{C} \right\}.$$

Since the first row determines these matrices, we might thing of $\mathbb{H}$ as $(u,v) \in \mathbb{C}^2$ with standard addition and multiplication given by

$$(u,v)(w,z) = (uw - \bar{z}v,\ zu + v\bar{w})$$

For $(u,v) \in \mathbb{H}$ let $\overline{(u,v)} := (\bar{u}, -v)$

then $(u,v) \cdot \overline{(u,v)} = (\underbrace{u\bar{u} + \bar{v}v}_{\substack{\text{real} \\ \geq 0}},\ 0)$

So $$(u,v)^{-1} = \frac{\overline{(u,v)}}{u\bar{u} + \bar{v}v}$$

if $(u,v) \neq (0,0)$.

We can do entirely the same calculation for the octonions
$\mathbb{O} = \{(a,b) : a, b \in \mathbb{H}\}$. defining the multiplication by exactly the same formula, paying attention to the order of multiplication of elements of $\mathbb{H}$, since $\mathbb{H}$ is not commutative

It turns out that $\mathbb{O}$ is not associative, but we have a multiplication with unit $\overline{}$

$(1,0)$ and inverse $(a,b)^{-1} = \dfrac{\overline{(a,b)}}{a\bar{a}+\bar{b}b}$

if $(a,b) \neq (0,0)$.

The proof of 8.2.a is now immediate using the projection

$$F_k(\mathbb{R}^{n+1}) \longrightarrow F_{k-2}(\mathbb{R}_2^{n+1})$$

$$(x_1, x_2, \dots, x_k) \longmapsto \left(0, 4, 4(x_2-x_1)^{-1}(x_3-x_1), \dots, 4(x_2-x_1)^{-1}(x_k-x_1)\right)$$

for $n=3$, we consider the x_i as quaternions

for $n=7$, we consider the x_i as octonions.

Or simply use the existence of inverses for non-zero elements

Using the fact that for both multiplications we have $\|a\cdot b\| = \|a\|\cdot\|b\|$, if $\|a\| := \sqrt{a\bar{a}}$ $(=\sqrt{\bar{a}a})$ we see that there are no zero-divisors so that multiplication by $(x_2-x_1)^{-1}$ is injective.

The proof of <u>8.2 b</u> is in two steps.

First we want to show that the hypothesis implies that $n=3,7$.

We follow Fadell-Husseini here:

$$\begin{array}{ccc} F_k(\mathbb{R}^{n+1}) & \hookleftarrow & F_{k-r}(\mathbb{R}^{n+1}) \\ \downarrow & & \\ F_r(\mathbb{R}^{n+1}) & & \end{array}$$

Assume a fibre homotopy equivalence

$$\begin{tikzcd} F_r(\mathbb{R}^{n+1}) \times F_{k-r}(\mathbb{R}^{n+1}_r) \arrow[rr, "\phi"] \arrow[dr, "\mathrm{proj}_1"'] & & F_k(\mathbb{R}^{n+1}) \arrow[dl, "p_{r,k}"] \\ & F_r(\mathbb{R}^{n+1}) & \end{tikzcd}$$

We may assume that the map ϕ on the fibre over the basepoint is htpc to the identity (otherwise precompose ϕ by

$\mathrm{id} \times \left(\text{homotopy inverse of } \phi\big|_{\text{fibre over } (q_1,\dots,q_r)}\right)$

The map $\alpha_{21}' : S^n \longrightarrow F_k(\mathbb{R}^{n+1})$ maps under

$p_{r,k}$ to $\alpha_{21}' : S^n \longrightarrow F_r(\mathbb{R}^{n+1})$ (recall that

$r \geq 2$), also the elements

$$\alpha_{st}, \quad r+1 \leq s \leq k, \quad 1 \leq t < s,$$

generate $\pi_n(F_{k-r}(\mathbb{R}^{n+1}_r))$ freely. We know

that in $\pi_{2n-1}(F_r(\mathbb{R}^{n+1}) \times F_{k-r}(\mathbb{R}^{n+1}_r))$ we have

$[\alpha_{21}, \alpha_{r+1\,1}] = 0$, and we have

$\pi_n(\phi)\,\alpha_{21} = \alpha_{21} + \sum_{s=r+1}^{k} \beta_s$, where

β_s is a linear combination of $\alpha_{s1}, \dots, \alpha_{s\,s-1}$

Therefore, we get an equation of the form

$$(1) \quad 0 = \left[\alpha_{21} + \sum_{s=r+1}^{k} \beta_s \, , \, \alpha_{r+1,1} \right] \text{ in } \pi_{2n-1}\left(F_k(\mathbb{R}^{n+1})\right).$$

Now Yang-Baxter implies

$$[\alpha_{r+1\,1} \, , \, \alpha_{21} + \alpha_{2\,r+1}] = 0 \qquad \text{i.e.}$$

$$[\alpha_{21}, \alpha_{r+1\,1}] = \pm [\alpha_{r+1\,1}, \alpha_{r+1\,2}]$$

which lies in the image of $\pi_{2n-1}(\mathbb{R}^{n+1}_r)$ in the direct sum decomposition.

$[\alpha_{r+1\,1}, \beta_{r+1}]$ clearly lie in the same summand.

For $s > r+1$ we have $[\alpha_{r+1,1}, \alpha_{si}] = 0$ unless $i = 1$ or $r+1$. But

$0 = [\alpha_{s1}, \alpha_{r+1\,s} + \alpha_{r+1\,1}]$ implies

$[\alpha_{s1}, \alpha_{r+1\,1}] = \pm [\alpha_{s1}, \alpha_{s\,r+1}]$ which is in the $\pi_{2n-1}(s_{s-1})'(\pi_{2n-1}(\mathbb{R}^{n+1}_{s-1}))$ and

$0 = [\alpha_{s\,r+1}, \alpha_{1s} + \alpha_{1\,r+1}]$ implies the same for $[\alpha_{r+1}, \alpha_{s\,r+1}]$

Since all these summands are distinct we get after expanding β_{r+1} as a linear combination of $\alpha_{r+1\,s}$, $1 \le s \le r$, the equation

(11)
$$[\alpha_{21}, \alpha_{r+1\,1}] + \sum_{s=1}^{r} c_s [\alpha_{r+1\,s}, \alpha_{r+1\,1}] = 0$$

where $\beta_{r+1} = \sum_{1}^{r} c_s \alpha_{r+1\,s}$.

We have (see above)

$$[\alpha_{21}, \alpha_{r+1\,1}] = (-1)^n [\alpha_{r+1\,1}, \alpha_{21}]$$

$$= (-1)^{n+1} [\alpha_{r+1\,1}, \alpha_{2\,r+1}] \quad \left(\text{since } 0 = [\alpha_{r+1\,1}, \alpha_{21} + \alpha_{2\,r+1}] \right)$$

$$= [\alpha_{r+1\,1}, \alpha_{r+1\,2}]$$

$$c_1 [\alpha_{r+1\,1}, \alpha_{r+1\,1}] + ((-1)^n + c_2)[\alpha_{r+1\,2}, \alpha_{r+1\,1}] + \sum_{s=3}^{r} c_s [\alpha_{r+1\,s}, \alpha_{r+1\,1}] = 0$$

Using our last problem sheet we obtain that $c_2 = (-1)^{n+1}$. Using the same argument for the pair α_{21}, $\alpha_{r+1\,2}$ we obtain $c_1 = (-1)^{n+1}$

But then

$[\alpha_{r+1\,1}, \alpha_{r+1\,1}] = 0$, i.e. in the summand corr. to $\pi_*(\mathbb{R}^{n+1}_r) = \pi_*(S^n_1 \vee \cdots \vee S^n_r)$

$0 = [\iota_1, \iota_1] \in \pi_{2n-1}(S^n_1)$ i.e.

There exists a commutative diagram

$$\begin{tikzcd} S^n \times S^n \arrow[dr, "\mu"] & \\ S^n \vee S^n \arrow[u, hook] \arrow[r, "{(\iota,\iota)}"'] & S^n \end{tikzcd}$$

so that S^n is an H-space with an honest 1.

It is a famous result of J. Adams that this implies that $n = 1, 3, 7$.

<u>Step 2</u>: Assume now that $n = 3, 7$.

We know from step 1, checking the various summands that

$$(-1)^{n+1}\beta_{r+1} = \alpha_{r+1\,1} + \alpha_{r+1\,2}$$

and that for all $2 < s \le r$ we have by naturality of $[\,,\,]$

$[\alpha_{21} + \beta_{r+1}, \alpha_{r+1\,s}] = 0$. Since for these s

$[\alpha_{21}, \alpha_{r+1\,s}] = 0$ (by the second Y-B relation), we get

$$[\alpha_{r+1\,1}, \alpha_{r+1\,s}] + [\alpha_{r+1\,2}, \alpha_{r+1\,s}] = 0$$

These elements being linearly independent (last weeks exercise) we get a contradiction to each of these summands being non-zero. Therefore there is no s with $2 < s \leq r$. $\square$