6.A

Correction to Lectures 5 and 6: Proposition 5.11

Thanks to Peter Patzt's prodding we discovered a real blooper in the formulation of Proposition 5.11 and again thanks to some input of Peter we have now a short and easy proof of this proposition.

First the blooper: we want S_k to act on $F_k(M)$, so that for $\tau, \sigma \in S_k$ we have $(\tau \cdot \sigma)(x) = \tau(\sigma(x))$, $x \in F_k(M)$. Thinking of x as embedding

$$x : \{1, \dots, k\} \longrightarrow M$$

we defined σx in Proposition 5.11 by $x \circ \sigma$ so that

$$\tau \cdot \sigma(x) = x \circ \tau \circ \sigma = \sigma(\tau(x)).$$

Thus: if we prove the proposition for transpositions $(t\, t{+}1)$ only, we will obtain for $\sigma = \tau_p \circ \dots \circ \tau_1$, where each τ_i is a transposition

$$\sigma \circ \alpha'_{rs} = \tau_1 \circ \tau_2 \circ \dots \circ \tau_p \circ \alpha'_{rs}$$

$$\simeq \alpha'_{\tau_1 \circ \dots \circ \tau_p r,\ \tau_1 \circ \dots \circ \tau_p s} = \alpha'_{\sigma^{-1} r, \sigma^{-1} s}$$

Thus to keep a nice equation (and since we like to write maps from the left) we define

$$\sigma x : \{1, \dots, k\} \longrightarrow M \quad \text{by}$$

$$x \circ \sigma^{-1}$$

Since $\sigma = \sigma^{-1}$ for transpositions, we have proved

5.11 with the correct action (not the one of 5.11) (the original).

<u>Second</u>: Peter suggested slightly different representatives α'_{rs} for α_{rs}; one that allows to do away with distinguishing the cases $r>s$, $s>r$.

Thus: fix $r\neq s$ with $1\leq r,s\leq k$ and set

$$\alpha'_{rs}(\xi) = (q_1,\dots,q_{r-1},\ q_s+\xi,\ q_{r+1},\dots,q_k)\ ,\quad \xi\in S^n$$

Main advantage: works properly for both cases and α'_{rs} new is homotopic to α'_{rs} old, so the α_{rs} do not change.

Now <u>Proposition 5.11</u> reads as follows:

Let $\sigma\in S_k$ act on $F_k(\mathbb{R}^{n+1})$ by

$$\sigma(x_1,\dots,x_k) = (x_{\sigma^{-1}(1)},\dots,x_{\sigma^{-1}(k)})$$

then $\pi_n(\sigma)(\alpha_{rs}) = \alpha_{\sigma r\,\sigma s}$

<u>Proof</u>: For any map $h: X\longrightarrow F_k(M)\subset \underbrace{M\times\cdots\times M}_{k}$ we denote the i-th component by h_i.

Then
$$(\sigma\circ\alpha'_{rs}(\xi))_i = \begin{cases} q_{\sigma^{-1}(i)} & ,\ \sigma(r)\neq i\\ q_s+\xi & ,\ \sigma(r)=i\end{cases}$$

we also note that

$$(\sigma\circ\alpha'_{rs}(\xi))_{\sigma(s)} = q_s$$

On the other hand

$$\left(\alpha'_{\sigma(r)\sigma(s)}\left(\xi\right)\right)_i = \begin{cases} q_i & , \quad i \neq \sigma(r) \\ q_{\sigma(s)} + \xi & , \quad i = \sigma(r) \end{cases}$$

we also note that

$$\left(\alpha'_{\sigma(r)\sigma(s)}(\xi)\right)_{\sigma(s)} = q_{\sigma(s)}$$

Denote $\alpha'_{\sigma(r)\sigma(s)}$ by f_0.

Then f_0 is homotopic to f_1 where

$$f_{t\,i} = f_{0\,i} + (1-t)(q_s - q_{\sigma(s)})$$

(i.e. we add in each component the same element of $\mathbb{R}^{n+1}$. Thus this is a homotopy of maps into $F_k(\mathbb{R}^{n+1})$)

This simple homotopy is due to Peter and makes the rest of the proof very simple.

Notice:

$$f_{1\,i}(\xi) = \begin{cases} q_i + q_s - q_{\sigma(s)} =: y_i & i \neq \sigma(r) \\ q_s - \xi & i = \sigma(r) \end{cases}$$

and $f_{1\,\sigma(s)}(\xi) = q_s$

Notice that y_i, $i \neq \sigma(r), \sigma(s)$, is in the complement of $q_s + B^{n+1}$, where B^{n+1} is the unit ball in $\mathbb{R}^{n+1}$

The same holds for $q_{\sigma^{-1}(i)} = \sigma \circ \alpha'_{rs}(\xi)_i$ if $i \neq \sigma(r), \sigma(s)$

Thus if we remove the points in positions $\sigma(r), \sigma(s)$ we obtain in both cases elements

of $F_{k-2}(\mathbb{R}^{n+1} \setminus (q_s + B^{n+1}))$.

Now $M := \mathbb{R}^{n+1} \setminus (q_s + B^{n+1})$ is a connected manifold of dimension ≥ 2 (we actually have $n+1 \geq 3$)

Thus by lecture 3 we know that

$$F_{k-2}(M) \text{ is pathconnected.}$$

So using a path from $(y_i)_{i \in \{1,\dots,k\} \setminus \{\sigma(r),\sigma(s)\}}$ in $F_{k-2}(\mathbb{R}^{n+1} \setminus (q_s + B^{n+1}))$ to $(q_{\sigma^{-1}(i)})_{i \in \{1,\dots,k\} \setminus \{\sigma(r),\sigma(s)\}}$

and keeping the $\sigma(r)$ and $\sigma(s)$ coordinate of f_1 constant we obtain a homotopy

from f_1 to $\sigma \circ \alpha'_{rs}$. $\square$

<u>Remark</u>: One further advantage of Peter's definition of α'_{rs} is that it fits better our understanding of α'_{rs} representing an element of $\pi_n(F_k(\mathbb{R}^{n+1}))$

These elements are represented by basepoint preserving map. Thus a "true" representative should map $e_1 \in S^n$ to $(q_1, \dots, q_k)$. Our map fits this in all positions apart from the r-th, where we have $q_s + 1$. We will see the advantage of this when we deal with $F_k(\mathbb{R}^2)$ where the basepoint cannot be avoided.