6. The homotopy Lie algebra of $F_k(\mathbb{R}^{n+1})$, $n \geq 2$
Part 3

<u>Recall:</u> $q_i \in \mathbb{R}^{n+1}$, $q_i = 4(i-1)e_1$, $\{e_1, \dots, e_{n+1}\}$ standard basis of $\mathbb{R}^{n+1}$

$(q_1, \dots, q_k)$ basepoint of $F_k(\mathbb{R}^{n+1})$

$$\mathbb{R}^{n+1}_i = \mathbb{R}^{n+1} \setminus \{q_1, \dots, q_i\}$$

$$F_{k-r}(\mathbb{R}^{n+1}_r) = \{(x_1, \dots, x_k) \in F_k(\mathbb{R}^{n+1}) : x_i = q_i,\ i = 1, \dots, r\}$$

For $1 \leq s < r \leq k$:

$\alpha'_{rs} : S^n \longrightarrow F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}) \subset F_k(\mathbb{R}^{n+1})$ defined by

$$\xi \longmapsto (q_1, \dots, q_{r-1}, q_s + \xi, q_r, \dots, q_{k-1})$$

$1 \leq r < s \leq k$

$$\alpha'_{rs} : S^n \to F_{k-r+1}(\mathbb{R}^{n+1}_{r-1}) \subset F_k(\mathbb{R}^{n+1})$$

$$\xi \mapsto (q_1, \dots, q_{r-1}, q_{s-1} + \xi, q_r, \dots, q_{k-1})$$

Let $\alpha_{rs} \in \pi_n(F_k(\mathbb{R}^{n+1}))$ be the homotopy class of α'_{rs}.

We had proved:

<u>5.9. Proposition:</u> The α_{rs}, $1 \leq s < r \leq k$, form an additiv basis for the free abelian group $\pi_n(F_k(\mathbb{R}^{n+1}))$.

Our aim today is to establish some relations among Whitehead products of elements of $\pi_n(F_k(\mathbb{R}^{n+1}))$.

To reduce notational efforts we first study the action of the symmetric group S_k on $\pi_n(F_k(\mathbb{R}^{n+1}))$.

recall: $\sigma \in S_k$ acts on $F_k(\mathbb{R}^{n+1})$ by

$$(x_1, \dots, x_k) \longmapsto (x_{\sigma 1}, \dots, x_{\sigma k}).$$

Clearly σ does not preserve the basepoint, but this is of no concern for $n \geq 2$: $\pi_1(F_k(\mathbb{R}^{n+1})) = 0$. Thus σ induces an isomorphism of $\pi_n(F_k(\mathbb{R}^{n+1}))$.

<u>6.1 Proposition</u> (old Prop. 5.11). For any $\sigma \in S_k$ we have

$$\pi_n(\sigma)(\alpha_{rs}) = \alpha_{\sigma r\, \sigma s}$$

<u>Proof</u> It suffices to prove this for transpositions $(t\ t{+}1)$, $1 \leq t < k$, where $(t\ t{+}1)$ exchanges t and $t+1$ and leaves everything else fixed.

There are a number of cases to consider:

(i) $\{s, r\} \cap \{t, t+1\} = \emptyset$.

α_{rs} maps ξ to $\left(q_1, \dots, q_{r-1}, \begin{array}{cc} q_s + \xi & s < r \\ q_{s-1} + \xi & s > r \end{array}, q_r, \dots, q_{k-1}\right)$

notice that for $s < r$ q_s appears in position s and for $s > r$ q_{s-1} appears in position s; and $q_s + \xi$ (or $q_{s-1} + \xi$) in position r.

These are untouched by $(t\ t{+}1)$. Thus it suffices to describe a homotopy that keeps all positions different from t and $t+1$ fixed and moves in position t q_i to q_j and in position $t+1$ q_j to q_i where q_i is $(t\ t+1)\alpha_{rs}(\xi)_t$ and q_j is $(t\ t+1)\alpha_{rs}(\xi)_{t+1}$.

t-th coordinate

(t+1)-st coordinate

We have the cases

→ (1a) $t+1 < s < r$: then $q_i = q_{t+1}$, $q_j = q_t$ we move q_{t+1} and q_t on the circle with center $\frac{1}{2}(q_t + q_{t+1})$ and radius 2 counterclockwise in the e_1-e_2-plane in their positions, so that at each time they are antipodal points. This homotopy is clearly in $F_k(\mathbb{R}^{n+1})$

(1b) $s < t < t+1 < r$ $q_i = q_{t+1}$, $q_j = q_t$; and we do the same

→ (1c) $s < r < t$ $q_i = q_t$, $q_j = q_{t-1}$

Now $t-1$ might be r; but again: no point in the positions different from t and $t+1$ will meet the semicircle from q_{t-1} to q_t with center $\frac{1}{2}(q_{t-1} + q_t)$

$\|q_s + \xi\| < |q_{t-1}|$, $|q_m| < |q_{t-1}|$, $m \leq r-1$

(1.d) $t+1 < r < s$ $q_i = q_{t+1}, q_j = q_t$

homotopy is ok

→ (1.e) $r < t < t+1 < s$ $q_i = t, \; q_j = t-1$

in position r we have $q_{s-1} + \xi$, and $s-1 > t$, so that again we have no colliding points

(1.f) $r < s < t$ $q_i = t, \; q_j = t-1$

Since $s-1 < t-1$, we again have no collisions.

(ii) $\{s,r\} \cap \{t, t+1\}$ consists of one point.

We only discuss two of the potentially many cases.

(a) $s < r = t$

$$(t\,t{+}1)\,\alpha'_{rs}(\xi) = (q_1, \dots, q_s, \dots, q_{r-1}, q_r, q_s + \xi, q_{r+1}, \dots, q_{k-1})$$

$$\alpha'_{\sigma r \sigma s} = \alpha'_{r+1\, s}$$

$$\alpha_{r+1\, s}(\xi) = (q_1, \dots, q_s, \dots, q_r, q_s + \xi, q_{r+2}, \dots, q_{k-1})$$

(b) $s = t < t+1 < r$. Then

$$\underset{s+1<r}{(t\ t+1)\,\alpha'_{rs}(\xi) = (s\ s+1)\,\alpha'_{rs}(\xi)} = (q_1, \dots, \overset{s}{q_{s+1}}, q_s, \dots q_{r-1}, \overset{r}{q_s+\xi}, q_r, \dots, q_{k-1})$$

and

$$\alpha_{r\,s+1}(\xi) = (q_1, \dots, \overset{s}{q_s}, \overset{s+1}{q_{s+1}}, \dots, q_{r-1}, \overset{r}{q_{s+1}+\xi}, q_r, \dots, q_{k-1})$$

with four homotopies we move $\alpha_{r\,s+1}$ to $(t\,t+1)\,\alpha_{rs}$

1st. Lift in positions $s+1$ and r

$$q_{s+1} \text{ to } q_{s+1} + 3e_2, \quad q_{s+1}+\xi \text{ to } q_{s+1}+\xi+3e_2$$

2nd Move in positions s $\quad q_s$ to q_{s+1}

3rd Move in positions $s+1$ and r

$$q_{s+1}+3e_2 \text{ to } q_s+3e_2 \qquad \text{and}$$

$$q_{s+1}+\xi+3e_2 \text{ to } q_s+\xi+3e_2$$

4th Move in positions $s+1$ and r

$$q_s+3e_2 \text{ to } q_s \quad \text{and } q_s+\xi+3e_2 \text{ to } q_s+\xi.$$

(iii) $\quad \{r,s\} = \{t, t+1\}$ $\qquad s<r \Rightarrow s=r-1 \quad s=t$

$\qquad s>r \Rightarrow s=r+1 \quad r=t$

Thus if $\quad (t\,t+1)\circ\alpha'_{rs} \simeq \alpha'_{sr}$ for $s<r$, then

$$\alpha'_{rs} = (t\,t{+}1)\circ(t\,t{+}1)\circ\alpha'_{rs}$$

$$\simeq (t\,t{+}1)\,\alpha'_{sr} \qquad \text{for } s<r$$

Exchanging r for s gives then

$$(t\,t{+}1)\,\alpha'_{rs} \simeq \alpha'_{sr} = \alpha'_{\sigma r\,\sigma s} \qquad \text{if } r<s$$

and $\{t\ t{+}1\} = \{r,s\}$.

So let $1 \le s < s{+}1 = r \le k$, $t = s$. Then

$$(t\,t{+}1)\,\alpha'_{s+1\,s}(\xi) = (s\ s{+}1)\,\alpha'_{s+1\,s}(\xi)$$

$$= \big(q_1, \dots, \underset{\downarrow}{\overset{s}{q_s+\xi}}, \overset{s+1}{q_s}, q_{s+1}, \dots, q_{k-1}\big)$$

$$= \alpha'_{s\,s+1}(\xi) \qquad \square$$

We know that any α_{rs} with $s>r$ is a $\mathbb{Z}$-linear combination of the α_{ij} with $j<r$. In fact,

6.2 Proposition: $\alpha_{rs} = (-1)^{n+1}\alpha_{sr}$ for $s \neq r$.

Proof. Because of 6.1 and the fact that any $\pi(\sigma)$, $\sigma \in S_n$, is an isomorphism, it suffices to prove that $\alpha_{21} = (-1)^{n+1}\alpha_{12}$

But

$$\alpha'_{21}(\xi) = (q_1, q_1+\xi, q_2, \dots, q_{k-1})$$

$$\alpha'_{12}(\xi) = (q_1+\xi, q_1, q_2, \dots, q_{k-1})$$

Consider the homeomorphism

$$h : F_k(\mathbb{R}^{n+1}) \longrightarrow \mathbb{R}^{n+1} \times F_{k-1}(\mathbb{R}^{n+1}_*)$$

$$(x_1, \dots, x_k) \longmapsto (x_1, \underbrace{(x_2-x_1, \dots, x_k-x_1)}_{=: h^2(x_1,\dots,x_k)})$$

Since $\mathbb{R}^{n+1}$ is contractible $\pi_n(h^2)$ is an isomorphism

$$h^2 \circ \alpha'_{21}(\xi) = (\xi, q_2-q_1, \dots, q_{k-1}-q_1)$$

$$h^2 \circ \alpha'_{12}(\xi) = (-\xi, q_2-q_1-\xi, \dots, q_{k-1}-q_1-\xi)$$

Consider $f_t(\xi) = (-\xi, q_2-q_1-(1-t)\xi, \dots, q_{k-1}-q_1-(1-t)\xi)$

This provides a homotopy between $h^2 \circ \alpha'_{12} = f_0$ and

$$f_1(\xi) = (-\xi, q_2-q_1, \dots, q_{k-1}-q_1)$$

But $-\mathrm{id} : \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R}^{n+1} - \{0\}$ induces in π_n multiplication by $(-1)^{n+1}$. This follows from the diagram

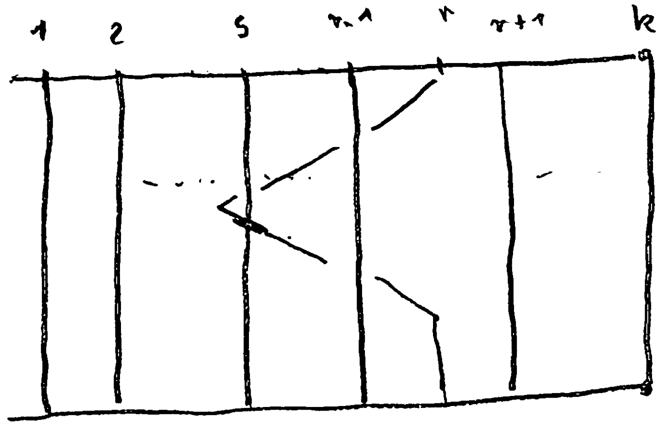
$$\begin{tikzcd} \mathbb{R}^{n+1}\setminus\{0\} \arrow[r, "-id"] & \mathbb{R}^{n+1}\setminus\{0\} \\ S^n \arrow[u, hook, "\simeq"] \arrow[r, "-id"] & S^n \arrow[u, hook] \end{tikzcd}$$

and the fact that $-id$ can be connected by a path in $O(n+1)$ to id if $n+1$ is even and to $\begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$ if $n+1$ is odd. $\square$

We are now ready for proving two sets of relations between Whitehead products of the α_{rs}

To motivate the first set we look at the situation in $F_k(\mathbb{R}^2)$. $\pi_1(F_k(\mathbb{R}^2))$ is the so called pure braid group, i.e we only consider braids whose strings connect the i-th point in $\mathbb{R}^2\times\{1\}$ to the i-th point in $\mathbb{R}^2\times\{0\}$, and we put the points in $\mathbb{R}^2$ into their standard position $4(i-1)\cdot e_1$.

Then α_{rs} corresponds to

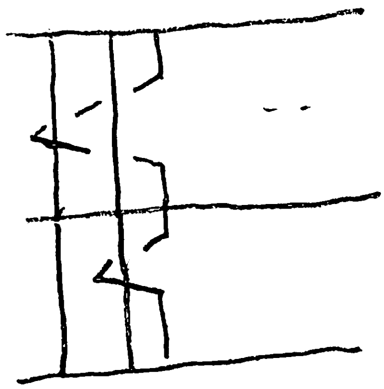


if we make some coherent choice to connect the α_{rs} to the basepoint

Now look at the commutator of

α_{21} and $\alpha_{31} \cdot \alpha_{32}$

$\alpha_{31} \cdot \alpha_{32}$ =



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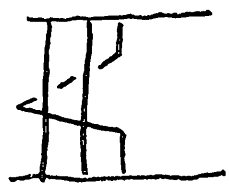
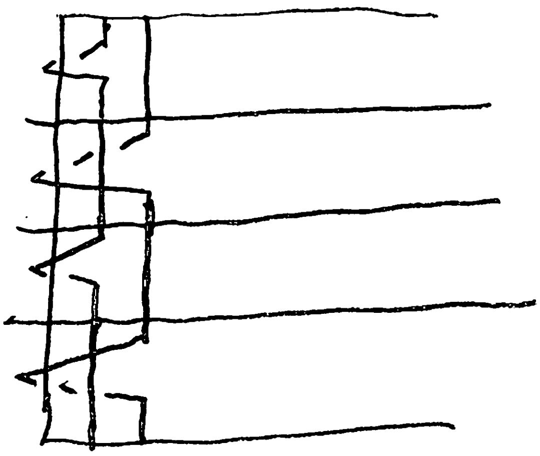


Thus $\alpha_{21} \cdot (\alpha_{31} \cdot \alpha_{32}) \cdot \alpha_{21}^{-1} \cdot (\alpha_{31} \cdot \alpha_{32})^{-1}$

=



= trivial, as you can easily see

This gives rise to the first set of relations in the following Theorem

<u>6.3 Theorem</u>: (Yang-Baxter relations for $F_k(\mathbb{R}^{n+1})$, $n \geq 2$)

For any $\sigma \in S_n$ we have the following relations

$$(i)\quad [\alpha_{\sigma(2)\sigma(1)}, \alpha_{\sigma(3)\sigma(1)} + \alpha_{\sigma(3)\sigma(2)}] = 0 \qquad k \geq 3$$

$$(ii)\quad [\alpha_{\sigma(2)\sigma(1)}, \alpha_{\sigma(4)\sigma(3)}] = 0 \qquad k \geq 4$$

<u>Proof</u>. Since Whitehead products are natural with respect to continuous maps by (6.1) we need only prove

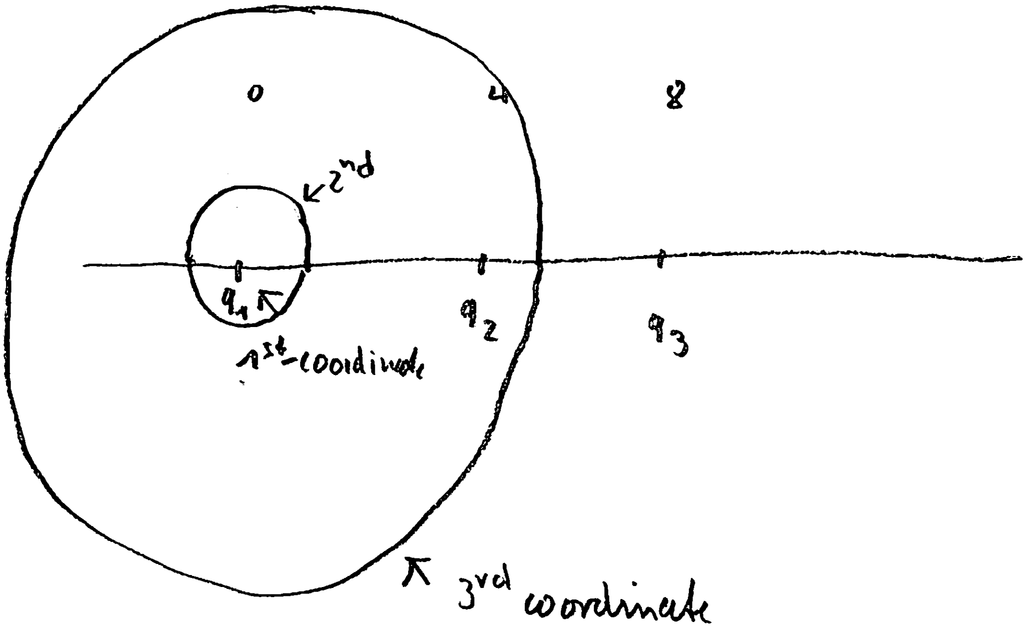
$$(i)'\ [\alpha_{21}, \alpha_{31} + \alpha_{32}] = 0$$

$$(ii)'\ [\alpha_{21}, \alpha_{43}] = 0$$

To prove $(i)'$ consider the map

$$f : S^n \times S^n \longrightarrow F_k(\mathbb{R}^{n+1})$$

$$(\xi_1, \xi_2) \longmapsto (q_1, q_1 + \xi_1, q_1 + 5\xi_2, q_3, \ldots q_{k-1})$$



Then: $f|\,S^n \times \{1\} \simeq \alpha'_{21}$ (not equal to α'_{21})

$f|\{1\}\times S^n \simeq \alpha'_{31} + \alpha'_{32}$

The Whitehead product $[\alpha_{21}, \alpha_{31} + \alpha_{32}]$ is the obstruction to extend $f|\,S^n \vee S^n : S^n \vee S^n \longrightarrow F_k(\mathbb{R}^{n+1})$ to $S^n \times S^n$, we get the first relation.

To prove (ii)' we proceed analogously. Take $f : S^n \times S^n \longrightarrow F_k(\mathbb{R}^{n+1})$ to be

$$f(\xi_1, \xi_2) = (q_1, q_1 + \xi_1, q_2, q_3 + \xi_2, q_4, \dots, q_{k-1})$$

Then $f|\,S^n \times \{1\} \simeq \alpha'_{21}$ $\quad f|\{1\}\times S^n \simeq \alpha'_{43}$

and we conclude as beforehand

For later use we state a special case of 6.3

6.4 In particular:

(i) $[\alpha_{32}, \alpha_{21} + \alpha_{31}] = 0$

(ii) $[\alpha_{31}, \alpha_{21} + (-1)^{n+1} \alpha_{32}] = 0$

Proof of (i) Use $\sigma = (1\ 3)$ to get

$$[\alpha_{32}, \alpha_{13} + \alpha_{12}] = 0 \qquad \text{and}$$

$$\alpha_{13} + \alpha_{12} = (-1)^{n+1} (\alpha_{31} + \alpha_{21})$$

Proof of (ii) Use $\sigma = (2\ 3)$ to obtain

$$[\alpha_{3,1}, \alpha_{21} + \underbrace{\alpha_{23}}_{= (-1)^{n+1} \alpha_{32}}] = 0$$

$\square$