4. The homotopy Lie algebra of $F_k(\mathbb{R}^{n+1})$, $n \geq 2$. Part 1

First some notation: Usually Lie algebras are associated with Lie groups. For us a <u>Lie algebra</u> (in degree 0) is

an abelian group L together with a <u>bilinear product</u>

$$[\ ,\ ] : L \times L \longrightarrow L \quad , \text{ called the Lie bracket}$$

such that

(i) $[X,Y] = -[Y,X]$ (antisymmetric) and

(ii) $[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$ (Jacobi identity)

Usually the Jacobi identity is written in the form

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

Homotopy and homology groups are graded groups (we will see shortly, how we grade the homotopy groups)

So we need to define the notion of a graded Lie algebra. This should be one satisfying the usual rules with the proviso that whenever an object of grade p passes an object of grade q a sign $(-1)^{pq}$ is introduced.

Thus:

<u>4.1</u> <u>Definition:</u> A <u>graded Lie algebra</u> is a graded abelian group $L = \bigoplus_{p \in \mathbb{Z}} L_p$ together

with a <u>bilinear</u> product

$$[\ ,\ ] : L \times L \longrightarrow L$$ s.t. for

$\lambda_p \in L_p$, $\mu_q \in L_q$, $\nu_r \in L_r$ we have:

$$[\lambda_p, \mu_q] \in L_{p+q}$$ and

(i) $[\lambda_p, \mu_q] = -(-1)^{p\cdot q}[\mu_q, \lambda_p]$ and

(ii) $[\lambda_p, [\mu_q, \nu_r]] = [[\lambda_p, \mu_q], \nu_r] + (-1)^{p\cdot q}[\mu_q, [\lambda_p, \nu_r]]$

$\square$

The standard example of a Lie-algebra with all elements in grade 0 is a ring R considered as an abelian group with Lie bracket $[r, s] := rs - s\cdot r$

The rôle of the Lie bracket in the case of homotopy groups of a space is the so called Whitehead product which we are going to define now:

Recall: If X and Y are finite CW-complexes $X \times Y$ is also a finite CW-complexes where the n-cells of $X \times Y$ are the products of p-cells of X and $(n-p)$-cells of Y, $p = 0, \dots, n$. If

$\chi_e : B^p \longrightarrow X$ is the characteristic map of the p-cell e^p of X and

$\chi_d : B^{n-p} \longrightarrow Y$ is the characteristic map of the $(n-p)$-cell d^{n-p} of Y then

$$\begin{array}{ccc} \chi_e\times\chi_d : B^p\times B^{n-p} & \longrightarrow & X\times Y \\ \cong & & \\ B^n & & \end{array}$$

is the characteristic map for $e^p\times d^{n-p}$.

Here we use the standard homeom. $B^n \xleftarrow{\cong} B^p\times B^{n-p}$

We think of S^p as a CW-complex with one 0-cell, the point $1=(1,0,\dots,0)$ and one p-cell where the characteristic map $B^p\longrightarrow S^p$ maps S^{p-1} to 1 and $B^p\setminus S^{p-1}$ homeomorphically to $S^p\setminus\{1\}$.

Then $S^p\times S^q$ consists of one 0-cell, a p-cell, a q-cell and a $(p+q)$-cell. The attaching map of any cell is the restriction of the characteristic map to the boundary sphere.

We assume that $p,q\geq 1$. Then the $\max(p,q)$-skeleton of $S^p\times S^q$ is $S^p\times 1\cup 1\times S^q$, i.e. the wedge $S^p\vee S^q$. Let $f_{p,q}: \partial(B^p\times B^q)\longrightarrow S^p\vee S^q$ be the attaching map of the final cell. Then

4.2 Definition: Let $\alpha \in \pi_p(X,x_0)$, $\beta \in \pi_q(X,x_0)$.

Define the Whitehead product $[\alpha,\beta] \in \pi_{p+q-1}(X,x_0)$ as the homotopy class of

$$S^{p+q-1} \xrightarrow{\cong} \partial(B^p \times B^q) \xrightarrow{f_{p,q}} S^p \vee S^q \xrightarrow{a+b} X$$

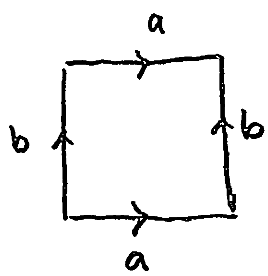
where $\alpha = [a]$, $\beta = [b]$, $a : S^p \to X$, $b : S^q \to X$

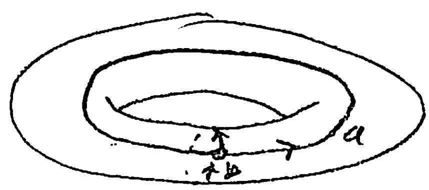
and $a+b$ is given by a on $S^p \times 1$ and b on $1 \times S^q$.

4.3 Proposition (without proof). $[\ ,\ ]$ is well-defined (i.e. independent of the choice of a and b)

4.4 Example: $p = q = 1$

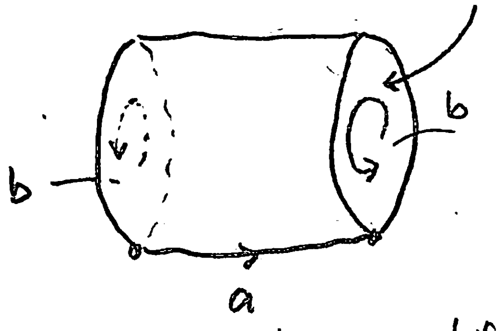
$S^1 \times S^1$ = torus





Thus $\big[[a],[b]\big] = [a][b][a]^{-1}[b^{-1}]$ (and this is obviously well defined)

$p = 1$, $q = 2$.



has opposite orientation as the one coming from the boundary of $B^3 = B^1 \times B^2$

the orientations of the 2-b's are different when regarded as the (oriented) boundary of $B^1 \times B^2$.

the map onto $S^1 \vee S^2$ is as follows

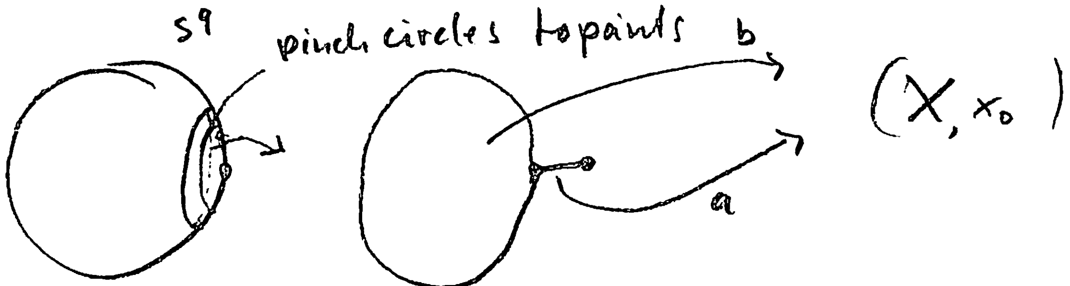
on $B^1 \times \partial B^2$ we use the characteristic map of e^1

on $\partial B^1 \times B^2$ the characteristic map of e^2; so up

to homotopy we get the element

$[b] - [a]\cdot[b]$, where $\pi_1(X, x_0)$ acts on

$\pi_q(X, x_0)$ as follows



(For $q=1$ it reduces to the previous picture).

Now: to get a graded Lie algebra there are two problems

1st: $\pi_1(X, x_0)$ is in general not abelian

2nd: Something is wrong with the grading.

<u>Solution for 1:</u> Assume $\pi_1(X, x_0)$ is abelian.

Then $[\ ,\ ]$ restricted to $\pi_1(X, x_0) \times \pi_1(X, x_0)$ is 0.

<u>Solution for 2:</u> Give $\pi_q(X, x_0)$ the grade (degree) $(q-1)$

Then $\pi_{p+q-1}(X, x_0)$ hat grade $p+q-2 = (p-1)+(q-1)$ ✓.

4.4 Remark. Recall the path fibration

$$P(X,x_0) \longrightarrow (X,x_0)$$

with fibre $\Omega(X,x_0)$. $P(X,x_0)$ is contractible and thus there is an isomorphism

$$\pi_i(X,x_0) \xrightarrow{\partial_i} \pi_{i-1}(\Omega(X,x_0), c_{x_0})$$

In $\Omega(X,x_0)$ we can multiply, so for

$$a : S^{p-1} \longrightarrow \Omega(X,x_0)$$
$$b : S^{q-1} \longrightarrow \Omega(X,x_0)$$

we can look at

$$\langle a,b\rangle : S^{p-1}\times S^{q-1} \longrightarrow \Omega(X,x_0)$$

$$(x,y) \longmapsto a(x)\,b(y)\,a(x)^{-1}\,b(y)^{-1}$$

with the proper model of $\Omega(X,x_0)$ this map is constant on $S^{p-1}\vee S^{q-1}$, and

$$S^{p-1}\times S^{q-1}/S^{p-1}\vee S^{q-1} \cong S^{p-1}\wedge S^{q-1} = S^{p+q-2}$$

and we get a map

$$S^{p+q-2} \longrightarrow \Omega(X,x_0)$$

One can pass to homotopy classes and thus gets a bracket $\langle\ ,\ \rangle : \pi_{p-1}(\Omega(X,x_0), c_{x_0})\times\pi_{q-1}(\Omega(X,x_0)) \to \pi_{p+q-2}(\Omega(X,x_0))$

and it is not too difficult to show that

$$\partial_{p+q-1}[\alpha,\beta] = \langle \partial_p \alpha, \partial_q \beta \rangle .$$

<u>4.5 Proposition</u> (without proof). If X is path connected, and $\pi_1(X)$ is abelian (this does not depend on the base point) then $(\pi_*(X,x_0), [\ ,\ ])$ is a graded Lie algebra.

Our need to resort to the Whitehead product operation on $F_k(\mathbb{R}^{n+1})$, $n \geq 2$, comes from the fact that the additive structure of $\pi_* F_k(\mathbb{R}^{n+1})$ carries very little information.

Look at our sequence of fibre bundles

$$\begin{array}{ccccc}
F_k(\mathbb{R}^{n+1}) & \longleftarrow & F_{k-1}(\mathbb{R}^{n+1} \setminus \{q_1\}) & \cdots\cdots & F_1(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_{k-1}\}) \\
\downarrow p_0 & & \downarrow p_1 & & \downarrow p_{k-1} = id \\
\mathbb{R}^{n+1} & & \mathbb{R}^{n+1} \setminus \{q_1\} & & \mathbb{R}^{n+1} \setminus \{q_1,\dots,q_{k-1}\}
\end{array}$$

each vertical map of the form $(x_1,\dots,x_{k-r}) \mapsto (x_1$

The total space in the r-th column is the fibre of the preceding map.

Furthermore, we have

4.6 Proposition: For each r with $0 \leq r < k$ there exists a continuous map

$$s_r : \mathbb{R}^{n+1} \setminus \{q_1, \dots, q_r\} \longrightarrow F_{k-r}(\mathbb{R}^{n+1} \setminus \{q_1, \dots, q_r\})$$

such that $\quad \pi_r \circ s_r = \mathrm{id}$

(Such a map s_r is called a section of π_r; for each point x of the base space it choose in a continuous way a point in the fibre over x)

Proof. For $r = k-1$ this is obvious since $\pi_{k-1} = \mathrm{id}$.

For $0 \leq r < k-2$ choose $r-k-1$ distinct points

$z_1, \dots, z_{r-k-1}$ on $S^n \subset \mathbb{R}^{n+1}$.

For $x \in \mathbb{R}^{n+1} \setminus \{q_1, \dots, q_r\}$ let $d_x = \min\limits_{1 \leq j \leq r} (\|x - q_j\|)$;

Set $d_x = 1$ if $r = 0$.

Then define $s_r(x) = \left(x, x + \frac{1}{2} d_x \cdot z_1, \dots, x + \frac{1}{2} d_x z_{r-k-1}\right)$.

$\square$

4.7 Corollary: For every p and every $0 \leq r \leq k-1$

$$\pi_p\left(F_{k-r}(\mathbb{R}^{n+1} \setminus \{q_1, \dots, q_r\})\right) \cong \pi_p\left(F_{k-r-1}(\mathbb{R}^{n+1} \setminus \{q_1, \dots, q_{r+1}\})\right)$$

$$\oplus \, \pi_p\left(\mathbb{R}^{n+1} \setminus \{q_1, \dots, q_r\}\right)$$

Proof.

Since π_1 of all spaces involved is trivial, we are dealing only with abelian groups.

Proposition 4.6 implies that all maps

$$\pi_q\big(F_{k-r}(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_r\})\big) \longrightarrow \pi_q\big(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_r\}\big)$$

in the exact homotopy sequence of the fibration p_r are surjective. Therefore,

$$\pi_{q-1}\big(F_{k-r-1}(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_{r+1}\})\big) \longrightarrow \pi_{q-1}\big(F_{k-r}(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_r\})\big)$$

is injective. Thus for every p we have a short exact sequence

$$0 \to \pi_p\big(F_{k-r-1}(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_{r+1}\})\big) \xrightarrow{\pi_p(\text{incl.})} \pi_p\big(F_{k-r}(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_r\})\big) \xrightarrow{\pi_p(p_r)} \pi_p\big(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_r\}\big) \to 0$$

The maps $\pi_p(s_r)$, $\pi_p(\text{incl.})$ then define a direct sum decomposition of the group in the middle. $\square$

Thus: looking at the additive structure of the homotopy groups there is no difference between the fibre bundle

$$F_{k-r}(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_r\}) \longrightarrow \mathbb{R}^{n+1} \setminus \{q_1,\dots,q_r\}$$

and the trivial bundle over $\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_r\}$ with fibre $F_{k-r-1}(\mathbb{R}^{n+1} \setminus \{q_1,\dots,q_{r+1}\})$. We will see later, that the Whitehead product will be helpful to distinguish these two bundles in most cases.

For the moment we simply note: $(n \geq 2)$

4.8 Let $n \geq 2$; then

$$\pi_p(F_k(\mathbb{R}^{n+1})) \cong \pi_p(F_{k-1}(\mathbb{R}^{n+1} \setminus \{q_1\}))$$

$$\cong \pi_p(\mathbb{R}^{n+1} \setminus \{q_1\}) \oplus \pi_p(F_{k-2}(\mathbb{R}^{n+1} \setminus \{q_1, q_2\})) \cong \cdots$$

$$\cong \prod_{j=1}^{k-1} \pi_p(\mathbb{R}^{n+1} \setminus \{q_1, \dots, q_j\}). \qquad \square$$

Notice that $\mathbb{R}^{n+1} \setminus \{q_1, \dots, q_j\}$ is homotopy equivalent to $\underbrace{S^n \vee \cdots \vee S^n}_{j \text{ summands}}$.

Therefore, we are interested in $\pi_p(S^n \vee \cdots \vee S^n)$ to get insight into the additive structure of $\pi_p(F_k(\mathbb{R}^{n+1}))$.