3. Basic Fibrations and the fundamental group

Recall from our first lecture

<u>1.1 Definition</u>. Let X be a topological space and $k \in \mathbb{N} \setminus \{0\}$

Then $F_k(X) := \{(x_1, \dots, x_k) \in X^k \mid x_i \neq x_j \text{ if } i \neq j\}$ is called the k-th ordered configuration space of X. $F_k(X)$ is topologized as a subspace of X^k. □

Points of X^k correspond bijectively to maps $\{1, \dots, k\} \longrightarrow X$, and points of $F_k(X)$ correspond to embeddings $\{1, \dots, k\} \hookrightarrow X$.

If $1 \leq r \leq k$ and $i : \{1, \dots, r\} \hookrightarrow \{1, \dots, k\}$ is an embedding we obtain a map

$$i^* : F_k(X) \longrightarrow F_r(X)$$

by precomposing an embedding $x : \{1, \dots, k\} \hookrightarrow X$ with i, i.e. $x \longmapsto x \circ i$.

i^* is the restriction of a map $X^k \longrightarrow X^r$,

$$(x_1, \dots, x_k) \longmapsto (y_1, \dots, y_r) \text{ with } y_n = x_{i(n)}, \; n = 1, \dots, r.$$

Therefore i^* is continuous.

<u>3.1 Notation</u>. Let $q = (q_1, \dots, q_k)$ be a basepoint of $F_k(X)$. Then $F_{k-r,r}(X) := F_{k-r}(X \setminus \{q_1, \dots, q_r\})$

This makes sense as long as the space X has at least k points and $0 \leq r \leq k$. Obviously, $F_{k,0}(X) = F_k(X)$. Notice also that $F_0(X)$ is a point.

3.2 For the remainder of this section we assume that M is a (non-empty) connected manifold of dimension ≥ 1

3.3 Theorem. Let $1 \leq r \leq k$ and $i : \{1, \dots, r\} \hookrightarrow \{1, \dots, k\}$ be an embedding. Then $i^* : F_k(M) \longrightarrow F_r(M)$ is a fibre bundle with fibre $F_{k-r,r}(M)$.

Remark: Note that $F_{k-r,r}(M)$ does not depend on the choice of basepoint $q = (q_1, \dots, q_k)$ since $M \setminus \{q_1, \dots, q_r\}$ and $M \setminus \{q_1', \dots, q_r'\}$ are homeomorphic as long as $\{q_1, \dots, q_r\}$ and $\{q_1', \dots, q_r'\}$ have the same cardinality.

For $M = \mathbb{R}$ or S^1 this is obvious. For $\dim M > 1$ use exercise 4 on problem sheet 2: if N is any connected manifold (of any dimension), and $x, y \in N$, then there is a homeomorphism $h : N \to N$ with $h(x) = y$. Then use: $M \setminus \{x_1, \dots, x_s\}$ is connected if $\dim M > 1$ and M is connected.

Proof. Let $(x_1, \dots, x_r) \in F_r(M)$, then $(i^*)^{-1}(x_1, \dots, x_r)$

$$= \left\{ (y_1, \dots y_k) \in F_k(M) \mid y_{i(j)} = x_j \ , j = 1, \dots, r \right\}$$

$$\underset{\text{homeo}}{\cong} F_{k-r}(M \setminus \{x_1, \dots, x_r\}) \underset{\text{see remark}}{\cong} F_{k-r,r}(M).$$

So it suffices to produce a neighborhood U of $(x_1, \dots, x_r) \in F_r(M)$ and a homeomorphism

$$g_{\mathcal{U}} : \mathcal{U} \times F_{k-r}(M \setminus \{x_1, \dots, x_r\}) \longrightarrow (i^*)^{-1}\mathcal{U}$$

such that $i^* \circ g_{\mathcal{U}}(u, y) = u$ for

$u \in \mathcal{U}$ and $y \in F_{k-r}(M \setminus \{x_1, \dots, x_r\})$.

Since M is a manifold, say of dimension d, and $x_i \neq x_j$ for $i \neq j$ we find neighborhoods U_i of x_i, $i = 1, \dots, r$, such that the closures $\overline{U}_i$ of the U_i are pairwise disjoint, and homeomorphic to the closed unit ball $B^d \subset \mathbb{R}^d$ under a homeomorphism $h_i : \overline{U}_i \longrightarrow B^d$ which maps U_i to the open unit ball V^d in $\mathbb{R}^d$ and x_i to 0.

For $\mathcal{U}$ we take $U_1 \times \dots \times U_r$ which is a neighborhood of $(x_1, \dots, x_r)$ in M^r. Since the U_i are disjoint $\mathcal{U}$ is contained in $F_r(M)$.

Next we construct a homeomorphism

$V^d \times B^d \xrightarrow{g} V^d \times B^d$ with

(i) $g(y, z) = (y, g_y(z))$

(ii) $g_y(y) = 0 \in \mathbb{R}^d$

(iii) $g_y(z) = z$ for $z \in S^{d-1} = B^d \setminus V^d$.

$$\alpha : V^d \longrightarrow \mathbb{R}^d \qquad \alpha(x) = \frac{x}{1-\|x\|} \qquad \text{and}$$

$$\beta : \mathbb{R}^d \longrightarrow V^d \qquad \beta(y) = \frac{y}{1+\|y\|}$$

are homeomorphisms with $\beta = \alpha^{-1}$.

Then set $\quad g_y(z) = \begin{cases} \beta(\alpha(z) - \alpha(y)) & z \in V^d \\ z & z \in S^{d-1} \end{cases}$

then obviously, (ii) and (iii) are satisfied. Furthermore for each $y \in V^d$ $\quad g_y : B^d \longrightarrow B^d$ is bijective, and a homeomorphism when restricted to V^d and S^{d-1}. We need to show that

$$\bar{g} : V^d \times B^d \longrightarrow B^d$$
$$(y, z) \longmapsto g_y(z)$$

is continuous in (y,z) with $\|z\| = 1$. This means: if $y_j \to y \in V^d$ and $z_j \longrightarrow z$ with $\|z\| = 1$, then

$$g_{y_j}(z_j) \longrightarrow z \, ,$$

and here it suffices to restrict to $z_j \in V^d$ for all j. Thus $\dfrac{y_j}{1-\|y_j\|}$ is bounded while $\dfrac{z_j}{1-\|z_j\|}$ goes to infinity, so that the second summand in

$$g_{y_j}(z_j) = \left(\frac{z_j}{1-\|z_j\|} - \frac{y_j}{1-\|y_j\|} \right) \left(\frac{1}{1 + \left\| \frac{z_j}{1-\|z_j\|} - \frac{y_j}{1-\|y_j\|} \right\|} \right)$$

converges to 0

while
$$\frac{z_j}{(1-\|z_j\|)\left(1+\left\|\frac{z_j}{1-\|z_j\|}-\frac{y_j}{1-\|y_j\|}\right\|\right)}$$

$$= \frac{z_j}{(1-\|z_j\|)\left(1+\frac{\|z_j\|}{1-\|z_j\|}+c_j\right)}$$

$$= \frac{z_j}{1+c_j(1-\|z_j\|)} \qquad \text{where } |c_j| < C .$$

Since $z_j \to z$ and $\|z\| = 1$, the first summand converges to z.

We use g, or rather $\bar{g} : V^d \times B^d \longrightarrow B^d$

to define a continuous map

$$U \times M \xrightarrow{\bar{h}_U} M \qquad \text{with the following properties}$$

(i) $h_{U,y} : M \longrightarrow M$ is a homeomorphism
$z \longmapsto \bar{h}_U(y,z)$ for each $y = (y_1, \dots, y_r) \in U$

(ii) $h_{U,y}(y) = x = (x_1, \dots, x_r)$ for each $y \in U$

(iii) $h_{U,y}(z) = z$ for $z \in M \setminus \bigcup_{j=1}^{r} U_j$.

To do this, simply define

$$h_{U,(y_1,\dots,y_r)}(z) := \begin{cases} h_j^{-1}\left(\bar{g}(\cdot\, h_j(y_j), h_j(z))\right) & ; z \in \overline{U}_j \\ z & ; z \notin \bigcup_j U_j \end{cases}$$

Since $h_j(x_j)=0$ the map $\bar h_u$ has property (ii)

Since each $g_Y : B^d \to B^d$ is a homeomorphism which restricts to the identity on S^{d-1} each $h_{U,Y}$ is a homeomorphism of $\bigcup_j \bar U_j$ which restricts to the identity on $\bigcup_j (\bar U_j \setminus U_j)$. Thus each $h_{u,y}$ is a homeom. of M satisfying (iii). Furthermore, $\bar h_u : U \times M \to M$ is continuous. Now define

$$h_u : (i^*)^{-1} U \longrightarrow U \times F_{k-r}(M \setminus \{x_1, \dots, x_r\})$$

by

$$h_u\big((z_1, \dots, z_k)\big) = \Big((z_{i(1)}, \dots, z_{i(r)}),$$

$$\big(\bar h_u((z_{i(1)}, \dots, z_{i(r)}), z_{j_1}), \dots,$$

$$\bar h_u((z_{i(1)}, \dots, z_{i(r)}), z_{j_{k-r}})\big)\Big)$$

where $j_1 < \dots < j_{k-r}$ and

$$\{z_{j_1}, \dots, z_{j_{k-r}}\} = \{z_1, \dots, z_k\} \setminus \{z_{i(1)}, \dots, z_{i(r)}\}$$

Since $h_{u,(z_{i(1)},\dots,z_{i(r)})} : M \longrightarrow M$

maps $z_{i(p)} \in U_p$ to x_p the map $h_{u,(z_{i(1)},\dots,z_{i(r)})}$ restricts

to a homeomorphism $M \setminus \{z_{i(1)},\dots,z_{i(r)}\} \to M \setminus \{x_1,\dots,x_r\}$

and thus h_U maps the fibre of $(i^*)^{-1}\left(z_{i(1)},\dots,z_{i(r)}\right)$

homeomorphically to $\{(z_{i(1)},\dots,z_{i(r)})\}\times F_{k-r}(M\setminus\{x_1,\dots,x_r\})$.

$\square$

Two applications:

<u>3.4 Theorem.</u> Let M be a connected manifold of dimension $d \geq 3$ and let $\pi_1(M)=0$. Then for all $k \geq 1$ $\pi_1(F_k(M)) = 0$.

<u>Proof</u>: Induction on k.

$k=1$: $F_1(M) = M$. ✓

$k \mapsto k+1$: consider the fibre bundle

$$F_{k+1}(M) \xrightarrow{i^*} F_k(M)$$

$$(x_1,\dots,x_{k+1}) \longmapsto (x_1,\dots,x_k)$$

its fibre is $F_1(M\setminus\{q_1,q_2,\dots,q_k\})$ and we have as portion of the exact sequence of the fibration i^* the exact sequence

$$\pi_1(F_1(M\setminus\{q_1,\dots,q_r\})) \to \pi_1 F_{k+1}(M) \to \pi_1 F_k(M)$$

Induction hypothesis gives: $\pi_1 F_k(M)$

Seifert-van Kampen theorem gives us

$$\pi_1(M \setminus \{q_1, \dots, q_r\}) \longrightarrow \pi_1(M) \quad \Big(\text{and } F_1(M \setminus \{q_1, \dots, q_r\}) = M \setminus \{q_1, \dots, q_r\}\Big)$$

is an isomorphism since $\dim M \geq 3$ (More details below). By exactness: $\pi_1 F_{k+1}(M) = 0$.

To show that the inclusion induces an isomorphism

$$\pi_1(M \setminus \{q_1, \dots, q_r\}) \longrightarrow \pi_1(M)$$

we use induction on r, starting with $r = 0$. Thus we only need to show:

If M is connected and $\dim M > 2$ then

$$\pi_1(M \setminus \{x\}) \longrightarrow \pi_1(M)$$

is an isomorphism.

For this consider a neighborhood U of x homeomorphic to $\mathbb{R}^d$, $d = \dim M$. Then $\{M \setminus x, U\}$ is an open cover of M with $M - \{x\}$, U and $U \cap (M \setminus \{x\}) = U - \{x\}$ connected. A special case of the Seifert-van Kampen theorem then says that

$$\pi_1(M - \{x\}) \longrightarrow \pi_1 M$$

is surjective with kernel the smallest normal subgroup of $\pi_1(M \setminus \{x\})$ containing the image of

$$\pi_1(U \setminus \{x\}) \longrightarrow \pi_1(M \setminus \{x\}).$$

But $U - \{x\} \cong \mathbb{R}^d - \{0\} \simeq S^{d-1}$ and $\pi_1 S^{d-1} = 0$ if $d - 1 \geq 2$.

<u>3.5 Remark</u>. The argument used to prove 3.4. applied to π_0 instead of π_1 shows:

If M is connected and $\dim M \geq 2$ then $F_k(M)$ is connected for all k (We should have actually proved this before proving 3.4) $\square$

Again the same argument, and using the fact that for any connected surface Σ different from S^2 and $\mathbb{R}P^2$ (real projective plane) we have $\pi_j(\Sigma) = 0$ for all $j > 1$, we obtain

<u>3.6 Theorem:</u> Let Σ be a connected surface different from S^2 and $\mathbb{R}P^2$. Then for all $k \geq 1$ and $j \geq 2$ we have $\pi_j(F_k \Sigma) = 0$. $\square$

<u>Remark</u>. A path-connected space X with

$\pi_j(X) = 0$ for all $j \geq 2$ is called <u>aspherical</u> or a $K(\pi, 1)$ with $\pi = \pi_1 X$.

Any two $K(\pi, 1)$ which are homotopy equivalent to a CW-complex are homotopy equivalent. In fact, if X, Y are aspherical CW-complexes, then for any homomorphism $\pi_1(X, x_0) \xrightarrow{\varphi} \pi_1(Y, y_0)$ there exists up to homotopy a unique map $f: (X, x_0) \to (Y, y_0)$ with $\pi_1(f) = \varphi$.