2. Basic topological terminology

2.1 Homotopy groups

n-th homotopy group of a space X with basepoint x_0 is as a set the set of homotopy classes of (continuous) maps $f: S^n \longrightarrow X$ mapping the basepoint $1=(1,0,\ldots,0) \in S^n \subset \mathbb{R}^{n+1}$ to x_0 and homotopies are required to preserve basepoints.

There are also relative versions; for a pair (X, A) with $x_0 \in A$ we consider maps

$$f: (D^n, S^{n-1}, 1) \longrightarrow (X, A, x_0)$$

of triples and homotopies of triples

$$F: (D^n \times I, S^{n-1} \times I, \{1\} \times I) \longrightarrow (X, A, x_0)$$

To explain addition in these sets it is convenient to consider maps of triples

$$f: (I^n, I^{n-1}, J^{n-1}) \longrightarrow (X, A, x_0)$$

Here we view $I^0 = \{0\} \subset I^1 = I = [0,1] \subset I^2 \subset \cdots$

exactly as $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots$

and set J^{n-1} = closure of $(\dot{I}^n - I_{n-1})$ where $\dot{I}^n$ is the boundary of the n-cube I^n, i.e.

$$\dot{I}^n := \{ (x_1, \ldots, x_n) \in I^n : \text{at least one of } x_1, \ldots, x_n \text{ lies in } \{0,1\} \}$$

2.1.a. There is a homeomorphism of triples

$$\left(I^n/J^{n-1},\ I^{n-1}/I^{n-1}\cap J^{n-1},\ J^{n-1}/J^{n-1} \right) \to (D^n, S^{n-1}, 1)$$

which induces a bijection

between continuous maps

$$(I^n, I^{n-1}, J^{n-1}) \longrightarrow (X, A, x_0)$$

and

$$(D^n, S^{n-1}, 1) \longrightarrow (X, A, x_0)$$

and the corresponding triple homotopy classes. So both sets are isomorphic

The corresponding set of homotopy classes will be denoted by

$$\pi_n(X, A, x_0)$$

Addition for $n \geq 2$. Consider

$$[f], [g] \in \pi_n(X, A, x_0)$$

$$f, g : (I^n, I^{n-1}, 1) \longrightarrow (X, A, x_0)$$

Set $f+g : (I^n, I^{n-1}, 1) \longrightarrow (X, A, x_0)$

by $(f+g)(x_1, \dots, x_n) = \begin{cases} f(2x_1, \dots, x_n) & 0 \leq x_1 \leq \frac{1}{2} \\ g(2x_1 - 1, \dots, x_n); & \frac{1}{2} \leq x_1 \leq 1 \end{cases}$

Since for $n \geq 2$ the points $(0, x_2, \dots, x_n)$ and $(1, x_2, \dots, x_n)$ are in J^{n-1}

$f+g$ is continuous and we

set $[f]+[g] := [f+g]$

2.1.b This is well defined and makes $\pi_n(X, A, x_0)$ into a group which is abelian for $n \geq 3$.

$\pi_1(X, A, x_0)$ is a set with a distinguished element, the homotopy class of the constant map $I \to \{x_0\} \subset X$.
(If $[f] \in \pi_1(X, A, x_0)$ then $f : I \to X$ with $f(0) \in A$, $f(1) = x_0$).

2.1.c $\pi_n(X,\{x_0\},x_0)$ is simply called the n-th ("absolute") homotopy group of X with basepoint x_0 and denoted by $\pi_n(X,x_0)$

Since for any $[f]\in\pi_n(X,x_0)$ we have $f(\dot{I}^n)=\{x_0\}$ the elements of $\pi_n(X,x_0)$ correspond to homotopy classes of maps of pairs

$$(S^n,1)\longrightarrow(X,x_0)$$

Recall: $I^n/\dot{I}^n \cong D^n/S^{n-1} \cong S^n$.

Here we also define an addition for $n=1$ by exactly the same formula as before

$$(f+g)(x_1)=\begin{cases} f(2x_1) & 0\le x_1\le \frac{1}{2} \\ g(2x_1-1) & \frac{1}{2}\le x_1\le 1 \end{cases}$$

$[f]+[g]:=[f+g]$ and obtain $\pi_1(X,x_0)$ the fundamental group of X.

It is in general not abelian, but $\pi_n(X,x_0)$ is abelian for $n\ge 2$.

$\pi_0(X,x_0)\cong\pi_0(X,A,x_0)$ is the set of path components of X (usually, nobody considers $\pi_0(X,A,x_0)$ since it does not depend on A) it has a distinguished element, the path component of x_0.

2.1.d Functoriality.

A map $f:(X,A,x_0)\to(X',A',x_0')$ of triples induces

a map $\pi_n(f):\pi_n(X,A,x_0)\longrightarrow\pi_n(X',A',x_0')$

mapping $[g]$, $g:(D^n,S^{n-1},1)\to(X,A,x_0)$

to $[f\circ g]\in\pi_n(X',A',x_0)$

$\pi_n(f)$ is well-defined and a group homomorphism if

$n\geq 2$ or ($n=1$ and $A=x_0$, $A'=x_0'$).

2.1.e Exact sequence of a pair.

Recall: a sequence $\cdots\to G_{n+1}\xrightarrow{f_{n+1}}G_n\xrightarrow{f_n}G_{n-1}\to\cdots$

of groups and homomorphisms is called exact if

for every n we have $\operatorname{im}(f_{n+1})=\ker f_n$.

For $n\geq 1$ we have a map

$$\pi_n(X,A,x_0)\xrightarrow{\partial_n}\pi_{n-1}(A,x_0)$$

$$\left[f:(D^n,S^{n-1},1)\to(X,A,x_0)\right]\mapsto\left[f|_{S^{n-1}}:(S^{n-1},1)\to(A,x_0)\right]$$

which is a homomorphism for $n\geq 2$ and maps the distinguished element of $\pi_1(X,A,x_0)$ to the distinguished element of $\pi_0(A,x_0)$.

This gives us a long sequence of groups (and sets at the end) and homomorphisms (and set map at the end)

$$\cdots\to\pi_{n+1}(X,A,x_0)\xrightarrow{\partial_{n+1}}\pi_n(A,x_0)\to\pi_n(X,x_0)\to\pi_n(X,A,x_0)\xrightarrow{\partial_n}\cdots$$

$$\cdots\to\pi_2(X,A,x_0)\to\pi_1(A,x_0)\to\pi_1(X,x_0)\to$$

$$\pi_1(X,A,x_0)\xrightarrow{\partial_1}\pi_0(A,x_0)\to\pi_0(X,x_0)$$

All arrows without name are induced by inclusions

The sequence is exact wherever it makes sense:

for example, if $[f] \in \pi_1(X, x_0)$ and $[f]$ considered as an element of $\pi_1(X, A, x_0)$ is the distinguished element, then $[f]$ is in the image of $\pi_1(A, x_0) \to \pi_1(X, x_0)$ e.t.c.

2.1.f Examples. In general homotopy groups are hard to compute. A few are simpler to obtain.

$$\pi_i(S^n, 1) = \{0\}, \qquad 0 \le i < n$$

$$\pi_n(S^n, 1) \cong \mathbb{Z}, \qquad n \geq 1.$$

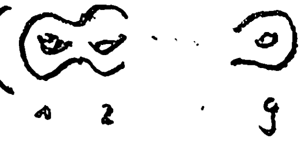
A representative for $[f] \in \pi_1(S^n, 1)$ corresponding to $k \in \mathbb{Z}$ is given by

$(z, x_2, \dots, x_n) \in \mathbb{C} \times \mathbb{R}^{n-1}$ with $z \cdot \bar{z} + x_2^2 + \dots + x_n^2 = 1$

is mapped by f to

$$\left(\frac{z^k}{\|z\|^{k-1}}, x_2, \dots, x_n \right) \qquad \text{with} \quad \frac{0^k}{\|0\|^{k-1}} := 0$$

For $n=1$, f wraps S^1 $|k|$-times around S^1 in positive $(k>0)$ or negative $(k<0)$ direction.

$\pi_1\Big($  $\Big) \cong \left\langle a_1, b_1, \dots, a_g, b_g : \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$

surface of genus g

Presentation of a group by generators and relations

$[a, b] := a b a^{-1} b^{-1}$.

2.2 Fibrations

2.2.a **Definition** Let $p: E \to B$ be a map (continuous as always)

(i) p has the homotopy lifting property with respect to X, if for any comm. diagram with solid lines the dotted map exists which makes the diagram commutative

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f_0"] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r, "g"] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

(ii) p is called a fibration if it has the homotopy lifting property (HLP) for any space X

(iii) p is called a Serre fibration if it has the homotopy lifting property for compact simplicial complexes (actually, it suffices to require that p has the HLP for all I^n, $n = 0, 1, 2, \dots$)

2.2.b Example. Let $PB = \{ w : [0,1] \to B : w(0) = b_0 \}$ for a fixed point $b_0 \in B$.

A subbasis for a topology on PB is given by the sets

$$W(K, O) := \{ w \in PB : w(K) \subset O \}$$ where $K \subset I$

is compact and $O \subset B$ is open.

Then
$$e : PB \longrightarrow B$$
$$w \longmapsto w(1)$$
is continuous and a fibration, the so called path-fibration of B. PB is contractible and is called the path space of (B, b_0)

2.2.c Exact sequence of a fibration.

Consider a (continuous) map $p : E \to B$, $b_0 \in B$ and $e_0 \in p^{-1}(b_0) =: F$. Then p induces a homomorphism

$$\pi_n(p) : \pi_n(E, F, e_0) \longrightarrow \pi_n(B, b_0)$$

<u>Claim:</u> If p is a Serre fibration then $\pi_n(p)$ is an isomorphism for $n \geq 2$, and a bijection for $n = 1$.

To prove for example surjectivity, consider a map of tripels

$$g : (I^n, I^{n-1}, J^{n-1}) \longrightarrow (B, \{b_0\}, b_0)$$

There is a homeomorphism $J^{n-1} \times [0,1] \xrightarrow{h} I^n$ such that $h(x,0) = x$, $x \in J^{n-1}$ (thus

$$h^{-1}(I^{n-1}) = \dot{J}^{n-1} \times [0,1] \cup J^{n-1} \times \{1\})$$

Now let $f_0 : J^{n-1} \times \{0\} \longrightarrow E$ be the constant map with value e_0 and $f : J^{n-1} \times [0,1] \longrightarrow E$ be the lift of $g \circ h$

$$\begin{tikzcd}
J^{n-1} \times \{0\} \arrow[r, "\text{constant}"] \arrow[d] & (E, F) \arrow[d, "p"] \\
J^{n-1} \times [0,1] \arrow[r, "g \circ h"] \arrow[ur, dashed, "f"] & (B, b_0)
\end{tikzcd}$$

Then $f \circ h^{-1}$ is a map $(I^n, I^{n-1}, J^{n-1}) \longrightarrow (E, F, b_0)$

and $p \circ f \circ h^{-1} = (g \circ h) \circ h^{-1} = g$.

Injectivity is similar

Putting 2.2.c and 2.1.e together, we get

<u>Theorem</u>: Let $p: E \to B$ be a Serre-fibration, $b_0 \in B$, $F := p^{-1}(b_0)$, $e_0 \in F$. Then for every $n \geq 1$ there exists a natural homomorphism (set-map for $n=1$)

$$\pi_n(B, b_0) \xrightarrow{\partial_n} \pi_{n-1}(F, e_0)$$ such that the sequence

$$\cdots \xrightarrow{\partial_{n+1}} \pi_n(F, e_0) \longrightarrow \pi_n(E, e_0) \xrightarrow{\pi_n p} \pi_n(B, b_0) \xrightarrow{\partial_n} \cdots$$

$$\longrightarrow \pi_1(E, e_0) \xrightarrow{\pi_1 p} \pi_1(B, b_0) \xrightarrow{\partial_1} \pi_0(F, e_0) \longrightarrow$$

$$\pi_0(E, e_0) \xrightarrow{\pi_0 p} \pi_0(B, b_0)$$

is exact

2.2.d Fibre bundles

<u>Definition</u>: A fibre bundle with fibre F is a continuous map $p: E \to B$ such that B has a covering $\mathcal{U}$ by open sets with the following property:

(*) for every $U \in \mathcal{U}$ there exists a homeomorphism

$$h_U : U \times F \longrightarrow p^{-1}(U) \qquad \text{s.t.}$$

$$p \circ h_U : U \times F \longrightarrow U \quad \text{is projection to the first factor.}$$

<u>Terminology</u>: $B \times F \xrightarrow{p_B} B$, $p_B(b, f) = b$, is called a trivial bundle (with fibre F).
p_B is also a fibration (Exercise). There are various

notions for morphisms between fibrations and fibre bundles. In general we have as a morphism between

$$\begin{array}{ccc} E & \text{and} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & & B' \end{array}$$

a pair of maps $f_E : E \to E'$, $f_B : B \to B'$ s.t

$$\begin{array}{ccc} E & \xrightarrow{f_E} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f_B} & B' \end{array}$$

commutes. Sometimes one associates to a space B the set of isomorphism classes of fibre bundles with fibre F over B. Then one requires f_B to be id_B.

2.2. e A fibre bundle need not be a fibration. If $p : E \to B$ and there is a locally finite partition of unity $\{\lambda_j\}_{j\in J}$ for B such that the covering $\{W_j\}_{j\in J}$, $W_j := \lambda_j^{-1}(0,1]$, satisfies property $(*)$ in Definition 2.2.d then p is a fibration.

[$\{\lambda_j\}_{j\in J}$, $\lambda_j : B \to [0,\infty)$, is a locally finite partition of unity if every $b \in B$ has a neighborhood U_b such that $\{j : \lambda_j^{-1}(0,\infty) \cap U_b \neq \emptyset\}$ is finite. Thus $\sum_{j\in J} \lambda_j$ is defined and the requirement is that $\sum \lambda_j = 1$]

<u>Corollary</u>: Every fibre bundle over a paracompact Hausdorff space is a fibration.