Energy solutions for the stochastic Burgers equation: Uniqueness and applications

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1 Introduction

The celebrated Kardar-Parisi-Zhang (KPZ) equation is the stochastic PDE

$$\partial_t h = \Delta h + |\partial_x h|^2 + \xi \tag{1}$$

for $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, where ξ is a *space-time white noise*, i.e. the centered Gaussian field $(\xi(\varphi))_{\varphi \in C_c^\infty(\mathbb{R}_+,\mathbb{R})}$ with covariance

$$\mathbb{E}[\xi(\varphi)\xi(\psi)] = \langle \varphi, \psi \rangle_{L^2}.$$

This is formally equivalent to $\mathbb{E}[\xi(s,x)\xi(t,y)] = \delta(t-s)\delta(x-y)$, where δ is the Dirac delta, although of course ξ cannot be evaluated pointwise. But this explains the name "white noise": The Fourier transform of the Dirac delta is the constant function 1. This means that ξ "excites all frequencies equally", just like white light is the superposition of all colors.

The KPZ equation appears in the mathematical description of growing interfaces [KPZ86], you might for example imagine the interface between a growing colony of bacteria and its environment. The problem is that the equation is badly ill posed: We expect that at a fixed time $t > 0$ the solution $h(t, \cdot)$ is a $1/2 - \kappa$ Hölder continuous function for any $\kappa > 0$, but not more regular than that (there is a simple powercounting argument to deduce this, but since regularity theory is not the focus of the lecture we will not discuss it). In particular h is far from being differentiable, and while $\partial_x h$ still makes sense as a Schwartz distribution, it is not clear how to even interpret the square $|\partial_x h|^2$.

There is however a cheap solution: Pretend for the moment that h is a smooth function and consider $w = \exp(h)$ (the so called Cole-Hopf transformation of h). Then

$$(\partial_t - \Delta)w = w(\partial_t - \Delta)h - w|\partial_x h|^2 = w\xi, \tag{2}$$

which is a perfectly well posed SPDE. Indeed, one way of obtaining a space-time white noise ξ is to start with a centered Gaussian process W indexed by the Borel sets $A \subset \mathbb{R}_+ \times \mathbb{R}_+$ with finite Lebesgue measure, such that $\mathrm{cov}(W(A)W(B)) = \lambda(A \cap B)$, where λ is the Lebesgue measure. For such a "Gaussian measure" we can use approximation by step functions to construct for any $\varphi \in L^2(\mathbb{R}_+, \mathbb{R})$ the integral $\int \varphi(t,x) W(\mathrm{d}t, \mathrm{d}x)$ as a centered Gaussian random variable, and it is easy to see that $\xi(\varphi) := \int \varphi(t,x) W(\mathrm{d}t, \mathrm{d}x)$ defines a space-time white noise. Moreover, using the martingale property of $t \mapsto \int \mathbb{1}_{[0,t]}(s) \varphi(s) W(\mathrm{d}s, \mathrm{d}x)$ and a suitable version of the Ito isometry we can extend the integral from deterministic integrands to "adapted" integrands. From here we can apply the variation of constants formula: If ξ was a smooth function, then we would expect that

$$w(t,x) = \int_{\mathbb{R}} p(t,y) w_0(y) \mathrm{d}y + \int_0^t \int_{\mathbb{R}} p(t-s, x-y) w(s,y) \xi(s,y) \mathrm{d}y \mathrm{d}s,$$

where

$$p(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-|x|^2/4t}$$

is the fundamental solution for the heat equation. Of course ξ is not a smooth function, but now we can interpret the integral involving ξ in terms of our stochastic integral and require that

$$w(t,x) = \int_{\mathbb{R}} p(t,y) w_0(y) \mathrm{d}y + \int_{[0,t]\times\mathbb{R}} p(t-s, x-y) w(s,y) W(\mathrm{d}s, \mathrm{d}y),$$

which is a perfectly well posed equation that can be solved by a Picard iteration. This is the approach of [Wal86].

Since $w = e^h$, we could then consider $\log w$ and see whether it solves the KPZ equation in any meaningful sense. There are two problems with that: First we need w to be strictly positive to even apply the logarithm. But for nonnegative non-zero initial conditions this is indeed the case, as was shown by [Mue91]. The second, much more substantial problem is that to write down the dynamics for $\log w$ we need $t \mapsto w(t,x)$ to be a semimartingale, otherwise we cannot apply Itô's formula. But this is not the case! While $t \mapsto w(t,x)$ is continuous, one can show that it has infinite quadratic variation. So while now we have a candidate solution h to the KPZ equation, we do not have any equation that we could write down for this candidate...

What we can do is to regularize the equation for w by mollifying the noise: Let $\rho \in C_c^\infty$ be a symmetric positive mollifier, i.e. $\rho \geqslant 0$ and $\int_{\mathbb{R}} \rho(x) \mathrm{d}x = 1$. Set $\rho^\varepsilon(x) = \varepsilon^{-1} \rho(\varepsilon^{-1} x)$ and (formally) $\xi^\varepsilon(t,x) := \int_{\mathbb{R}} \rho^\varepsilon(x-y) \xi(t,y) \mathrm{d}y$. Then the equation

$$\partial_t w^\varepsilon = \Delta w^\varepsilon + w^\varepsilon \xi^\varepsilon, \qquad w^\varepsilon(0) = w_0$$

also has a unique solution, and now the solution is infinitely smooth in the space variable, a semimartingale in the time variable, and it solves the equation classically (without going to the variation of constants formula). Moreover, if $w_0 \geqslant 0$ and $w_0 \neq 0$, then $w^\varepsilon(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$ and therefore we can apply Itô's formula to $h^\varepsilon = \log w^\varepsilon$:

$$\begin{aligned}
\mathrm{d}h^\varepsilon(t,x) &= \frac{1}{w^\varepsilon(t,x)} \mathrm{d}w^\varepsilon(t,x) - \frac{1}{2} \frac{1}{w^\varepsilon(t,x)^2} \mathrm{d}\langle w^\varepsilon(\cdot, x) \rangle_t \\
&= \frac{\Delta w^\varepsilon(t,x)}{w^\varepsilon(t,x)} \mathrm{d}t + \xi^\varepsilon(t,x) \mathrm{d}t - \frac{1}{2} \|\rho^\varepsilon\|_{L^2}^2 \mathrm{d}t \\
&= \left\{ \Delta h^\varepsilon(t,x) + |\partial_x h^\varepsilon(t,x)|^2 - \frac{1}{2\varepsilon} \|\rho\|_{L^2}^2 \right\} \mathrm{d}t + \xi^\varepsilon(t,x) \mathrm{d}t,
\end{aligned}$$

where we used formal notation $\xi^\varepsilon(t,x)\mathrm{d}t$ to denote the martingale term. Bertini and Giacomin [BG97] showed that for $\varepsilon \to 0$, w^ε converges to w, so at least formally $h = \log w$ solves

$$\partial_t h = \Delta h + |\partial_x h|^2 - \infty + \xi,$$

where $-\infty$ denotes the limit of $\frac{1}{2\varepsilon}\|\rho\|_{L^2}^2$ for $\varepsilon \to 0$. So if we want to solve the equation for h, we need to consider a renormalization. But actually it is still very unclear what the equation for h is, we only showed that h is the limit of solutions h^ε to "renormalized approximate equations".

It took 25 years from the introduction of the equation by the physicists Kardar, Parisi and Zhang until it could be mathematically rigorously formulated for the first time by Hairer [Hai13]. Hairer analyzed the regularity of the equation very carefully and performed an asymptotic expansion (for small times) of the solution around the solution to the linearized equation. He then derived an explicit equation for the remainder in this asymptotic equation (which is more regular than the solution itself) and solved this equation with the help of rough path integrals. This approach is nearly completely deterministic, it applies to a wide class of equations, and it lead to the new field of singular SPDEs and new techniques such as regularity structures [Hai14] or paracontrolled distributions [GIP15]. In these lectures I want to present an alternative, probabilistc approach that only applies to the stationary KPZ equation and which is very powerful for deriving this equation as scaling limit of interacting particle systems.

2 Energy solutions to Burgers equation

Everything in this section goes back to [GJ14, GJ13, GP18] .

2.1 Definition and basic properties

Energy solutions provide a probabilistic notion of solution to the KPZ equation and they go back to Gonçalves, Jara, and Gubinelli [GJ14, GJ13]. Since they are based on stationarity and since the invariant measures of the KPZ equation have infinite mass (the measures $\int_{\mathbb{R}} \mathbb{W}_{y,\lambda}\mathrm{d}y$, where $\mathbb{W}_{y,\lambda} = \mathrm{law}\left(\left(y + \sqrt{1/2}B_x + \lambda x\right)_{x\in\mathbb{R}}\right)$ for a two-sided Brownian motion B are invariant), it is easier to work instead with the derivative $u = \partial_x h$ of the KPZ equation. This derivative formally solves the conservative stochastic Burgers equation (in the following simply "Burgers equation")

$$\partial_t u = \Delta u + \partial_x u^2 + \partial_x \xi. \tag{3}$$

Indeed, since the derivative $\partial_x\left(y + \sqrt{1/2}B_x + \lambda x\right)$ does not depend on the starting point y, we would expect that u is invariant under the law of $\sqrt{1/2}\eta + \lambda$, where η is a white noise on $\mathbb{R}$, i.e. the derivative of the Brownian motion (exercise: convince yourself that the distributional derivative of the Brownian motion defines indeed a white noise).

We will take this formal observation as the starting point of our definition of (stationary) energy solutions. For simplicity we will always take $\lambda = 0$, and in these notes we will work with the equation on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, i.e. formally $u : \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$, although everything extends to $\mathbb{R}$ and this only makes some technical details slightly more complicated. Also, to simplify our life we want to get rid of the factor $\sqrt{1/2}$ in front of the white noise. We can do that by changing the parameters in the equation and considering instead

$$\partial_t u = \Delta u + \partial_x u^2 + \sqrt{2}\partial_x \xi, \tag{4}$$

which is formally invariant under the law of η. For simplicity we also restrict our attention to (probabilistically) weak solutions, although the arguments below actually also allow us to deal with strong solutions.

In the following we write $\mathcal{S}' = \mathcal{S}'(\mathbb{T})$ for the space of Schwartz distributions on $\mathbb{T}$, i.e. the dual space of $C^\infty(\mathbb{T})$. The difficulty in defining solutions to (4) is that u_t will only be a distribution and not a function, so there is no canonical way to make sense of the nonlinearity $\partial_x u_t^2$. A cheap way out is to define $\partial_x u^2$ through a limiting procedure. We say that ρ is a mollifier if $\rho \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\int \rho(x)\mathrm{d}x = 1$. In that case we define the rescaled mollifier $\rho^n(x) = n\rho(nx)$, and for $u \in \mathcal{S}'(\mathbb{T})$ we set

$$(u * \rho)(x) := \sum_{k\in\mathbb{Z}} u(\rho(x + k - \cdot)),$$

which defines an element of $C^\infty(\mathbb{T})$. One can also show that $u = \lim_{n\to\infty} u * \rho^n$ for any mollifier ρ. Before we get to our first definition, note that if ξ is a space-time white noise and $\varphi \in C^\infty(\mathbb{T})$, then

$$\int_0^t (\partial_x \xi)(s, \varphi)\mathrm{d}s := \xi(\mathbb{1}_{[0,t]} \otimes (-\partial_x \varphi)), \qquad t \geqslant 0,$$

is a continuous martingale with quadratic variation $\|\partial_x \varphi\|_{L^2}^2 t$.

Definition 1 *A <u>martingale solution</u> to the conservative stochastic Burgers equation (4) is a stochastic process u with trajectories in $C(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}))$ for which there exists a mollifier ρ such that for all $\varphi \in C^\infty(\mathbb{T})$ the process*

$$\begin{aligned} M_t(\varphi) &= u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\Delta\varphi)\mathrm{d}s - \lim_{n\to\infty} \int_0^t \int_{\mathbb{T}} \partial_x (u_s * \rho^n)^2(x)\varphi(x)\mathrm{d}x\mathrm{d}s \\ &= u_t(\varphi) - u_0(\varphi) - \int_0^t u_s(\Delta\varphi)\mathrm{d}s + \lim_{n\to\infty} \int_0^t (u_s * \rho^n)^2(\partial_x\varphi)\mathrm{d}s \end{aligned}$$

is a continuous martingale with quadratic variation $\langle M(\varphi)\rangle_t = 2\|\partial_x \varphi\|_{L^2}^2 t$. We say that u is a <u>stationary martingale solution</u> if for all $t \geqslant 0$ the map $\varphi \mapsto u_t(\varphi)$ defines a white noise.

It is relatively easy to see that martingale solutions exist, at least stationary ones. The challenge is to prove uniqueness (in law), which is difficult because we have virtually no control of the nonlinear part of the dynamics. On the state space of u the nonlinearity is not continuous, let alone locally Lipschitz-continuous, and therefore the usual arguments break down. But recall that u has a very specific structure because $u = \partial_x h$ and $h = \log w$ and the equation for w is well posed. So we could try to map two martingale solutions u_1 and u_2 to w_1 and w_2 and to use the well-posedness of the equation for w to deduce that w_1 and w_2 have the same law, and thus u_1 and u_2 have the same law.

But we already discussed above that we need a suitable Itô formula to carry out this program. And it seems hopeless to derive such a formula for martingale solutions, because we know nothing about the behavior of $\lim_{n\to\infty} \int_0^t (u_s * \rho^n)^2(\partial_x\varphi)\mathrm{d}s$ as a function of t, and in particular $t \mapsto u_t(\varphi)$ is not necessarily a semimartingale. So we need to refine our definition.

Recall that a stochastic process A with trajectories in $C(\mathbb{R}_+, \mathbb{R})$ is of zero quadratic variation if

$$\lim_{n\to\infty} \sum_{t_j \in \pi_n} |A_{t_{j+1}\wedge t} - A_{t_j \wedge t}|^2 = 0$$

whenever (π_n) is a sequence of partitions of $\mathbb{R}_+$ such that the mesh size (locally) converges to zero, and the convergence is uniformly on compacts in probability. In that case we also say that A has "zero energy". The following definition is due to [GoncalvesJara] (in slightly modified form).

Definition 2 *A martingale solution to (4) is called an energy solution if for all $\varphi \in C^\infty(\mathbb{T})$ the process*

$$A_t(\varphi) := -\lim_{n\to\infty} \int_0^t (u_s * \rho^n)^2 (\partial_x \varphi) \mathrm{d}s, \qquad t \geqslant 0,$$

has zero quadratic variation. Similarly we define stationary energy solutions.

It might seem more natural to require that $t \mapsto A_t(\varphi)$ has finite variation: After all, it should represent a drift term. But the spatial irregularity is so bad for this term that "we need to trade a bit of time regularity in order to control it in the space variable" (we will understand later what is meant by that), and it seems impossible to construct martingale solutions for which $A(\varphi)$ is of finite variation. But this is no problem: If $A(\varphi)$ has zero quadratic variation, then $u(\varphi)$ is a Dirichlet process, i.e. the sum of a local martingale and a zero quadratic variation process. And as Föllmer [Föl81] showed when he introduced his "pathwise Itô calculus", for Dirichlet processes we still have the same Itô formula as for semimartingales.

Of course, this only gives us an Itô formula for $u(\varphi)$ and not for $u(x)$ (which is not even well defined). So we need to mollify u, apply the Cole-Hopf transformation to it, and take the mollification away, hoping to show that the limit solves the stochastic heat equation (2). Let therefore

$$u_t^n(x) := (u_t * \rho^n)(x) = u_t\left(\overline{\rho_x^n}\right),$$

where $\overline{\rho^n}(y) = \sum_{k\in\mathbb{Z}} \rho^n(y+k)$ is the periodization of ρ^n and where we write

$$f_x(y) := f(x-y)$$

for all $f : \mathbb{T} \to \mathbb{R}$. The process u^n has trajectories in $C(\mathbb{R}_+, C^\infty(\mathbb{T}))$ and we can apply Itô's formula to it. But to obtain a candidate h^n for an approximate solution to the KPZ equation, we should integrate u^n in the space variable. On the torus there is a quite canonical way of constructing an anti-derivative: We define the Fourier transform of $u \in \mathcal{S}'(\mathbb{T})$ as

$$\hat{u}(k) := \mathcal{F}u(k) := u(e^{-2\pi i k \cdot}),$$

which can be shown to grow at most polynomially in k. If conversely $f : \mathbb{Z} \to \mathbb{C}$ is a sequence of numbers that grows at most polynomially, then we set

$$\mathcal{F}^{-1} f(\varphi) := \sum_{k\in\mathbb{Z}} f(k) \int e^{2\pi i k x} \varphi(x) \mathrm{d}x,$$

which is well defined because with integration by parts we see that

$$\sup_k \left| (2\pi i k)^\alpha \int e^{2\pi i k x} \varphi(x) \mathrm{d}x \right| = \sup_k \left| \int \partial^\alpha e^{2\pi i k x} \varphi(x) \mathrm{d}x \right| \leqslant \|\partial^\alpha \varphi\|_{L^1} < \infty$$

for all $\alpha \in \mathbb{N}_0^d$. Since $\mathcal{F}(\partial_x u)(k) = 2\pi i k \hat{u}(k)$, we set for $u \in \mathcal{S}'(\mathbb{T})$

$$Iu = \mathcal{F}^{-1}\left(\frac{\mathbb{1}_{k\neq 0}}{2\pi i k}\hat{u}\right),$$

which satisfies $\partial_x I u = \mathcal{F}^{-1}(\mathbb{1}_{k\neq 0}\hat{u}) = u - \int_{\mathbb{T}} u(x)\mathrm{d}x$ by definition. We then define

$$h_t^n(x) := I u_t^n(x) = u_t\left(I_x \overline{\rho_x^n}\right) = u_t(\Theta_x^n),$$

where

$$\Theta^n(x) = \sum_{k\neq 0} \frac{e^{2\pi i k x}}{2\pi i k}\mathcal{F}\overline{\rho^n}(k)$$

satisfies $\partial_x \Theta^n = \overline{\rho^n} - \mathcal{F}\overline{\rho^n}(0) = \overline{\rho^n} - 1$ and $\int_{\mathbb{T}} \Theta^n(x)\mathrm{d}x = 0$. In particular, we have

$$\mathrm{d}h_t^n(x) = u_t(\Delta\Theta_x^n)\mathrm{d}t + \mathrm{d}A_t(\Theta_x^n) + \mathrm{d}M_t(\Theta_x^n),$$

and since $\Delta_y \Theta_x^n(y) = \Delta_y \Theta^n(x-y) = \Delta_x \Theta_x^n(y)$ we get

$$\mathrm{d}h_t^n(x) = \Delta h_t^n(x)\mathrm{d}t + \mathrm{d}A_t(\Theta_x^n) + \mathrm{d}M_t(\Theta_x^n),$$

where $M(\Theta_x^n)$ is continuous martingale with quadratic variation

$$\mathrm{d}\langle M(\Theta_x^n)\rangle_t = \mathrm{d}\langle h^n(x)\rangle_t = 2\|\partial_y \Theta_x^n\|_{L^2}^2 \mathrm{d}t = 2\left\|\overline{\rho_x^n} - 1\right\|_{L^2}^2 \mathrm{d}t = 2\left\|\overline{\rho^n} - 1\right\|_{L^2}^2 \mathrm{d}t.$$

Note that by definition of Θ^n we would expect $\lim_{n\to\infty} h_t^n = \tilde{h}_t := h_t - \int_{\mathbb{T}} h_t(y)\mathrm{d}y$, where h solves the KPZ equation

$$\begin{aligned} h_t(\varphi) &= h_0(\varphi) + \int_0^t h_s(\Delta\varphi)\mathrm{d}s + \lim_{n\to\infty}\int_0^t \left(|\partial_x h * \rho^n|^2 - \|\overline{\rho^n}\|_{L^2}^2\right)(\varphi)\mathrm{d}s + N_t(\varphi) \\ &=: h_0(\varphi) + \int_0^t h_s(\Delta\varphi)\mathrm{d}s + \int_0^t (\partial_x h_s)^{\diamond 2}(\varphi)\mathrm{d}s + N_t(\varphi) \end{aligned}$$

where $\|\overline{\rho^n}\|_{L^2}^2$ is a renormalization constant that we already encountered when applying Ito's formula to the solution w of the stochastic heat equation, where $N(\varphi)$ is a continuous martingale with quadratic variation $2t\|\varphi\|_{L^2}^2$, and where $(\partial_x h_s)^{\diamond 2}$ denotes a renormalized square. Thus, we would expect that $\tilde{w}_t := e^{\tilde{h}_t} = w_t \exp\left(\int_{\mathbb{T}} h_t(y)\mathrm{d}y\right)$ satisfies a slight perturbation of the stochastic heat equation (2) (note that the process $\exp\left(\int_{\mathbb{T}} h_t(y)\mathrm{d}y\right)$ does not depend on the space variable!). So let us proceed and apply Ito's formula to $w_t^n(x) := \exp(h_t^n(x))$:

$$\begin{aligned} \mathrm{d}w_t^n(x) &= w_t^n(x)\mathrm{d}h_t^n(x) + \frac{1}{2}w_t^n(x)\mathrm{d}\langle h^n(x)\rangle_t \\ &= w_t^n(x)\left(\Delta h_t^n(x)\mathrm{d}t + \mathrm{d}A_t(\Theta_x^n) + \mathrm{d}M_t(\Theta_x^n) + \|\overline{\rho^n} - 1\|_{L^2}^2\,\mathrm{d}t\right) \\ &= \Delta w_t^n(x)\mathrm{d}t + w_t^n(x)\mathrm{d}M_t(\Theta_x^n) + w_t^n(x)\left(-\left(|\partial_x h_t^n(x)|^2 - \|\overline{\rho^n} - 1\|_{L^2}^2\right)\mathrm{d}t + \mathrm{d}A_t(\Theta_x^n)\right) \\ &= \Delta w_t^n(x)\mathrm{d}t + w_t^n(x)\mathrm{d}M_t(\Theta_x^n) + w_t^n(x)\left(-\left(\left|u_t\left(\overline{\rho_x^n} - 1\right)\right|^2 - \|\overline{\rho^n} - 1\|_{L^2}^2\right)\mathrm{d}t + \mathrm{d}A_t(\Theta_x^n)\right), \end{aligned}$$

where we used that h^n and w^n are smooth functions in x and therefore the chain rule yields $\Delta w^n = w^n \Delta h^n + w^n|\partial_x h^n|^2$. On a purely formal level this does not look bad: Formally

$$A_t(\Theta_x^n) = -\int_0^t u_s^2(\partial_y \Theta_x^n)\mathrm{d}s = \int_0^t u_s^2(\partial_x \Theta_x^n)\mathrm{d}s = \int_0^t u_s^2\left(\overline{\rho_x^n} - 1\right)\mathrm{d}s \to \int_0^t \left(u_s^2(x) - \int_{\mathbb{T}} u_s^2(y)\mathrm{d}y\right)\mathrm{d}s,$$

and since $\int_{\mathbb{T}} \overline{\rho^n}(y)\mathrm{d}y = 1$ we get

$$
\begin{aligned}
-\int_0^t \left(|u_s(\overline{\rho^n_x} - 1)|^2 - \|\overline{\rho^n} - 1\|_{L^2}^2 \right) \mathrm{d}s = & -\int_0^t \left(u_s(\overline{\rho^n_x})^2 - \int_{\mathbb{T}} u_s(\overline{\rho^n}(y))^2 \mathrm{d}y \right) \\
& - \int_0^t \left(-2u_s(\overline{\rho^n_x}) + \int_y u_s(y)\mathrm{d}y + 1 \right) \mathrm{d}s \\
& - \int_0^t \left(\int_{\mathbb{T}} u_s(\overline{\rho^n}(y))^2 \mathrm{d}y - \|\overline{\rho^n}\|_{L^2}^2 \right) \mathrm{d}s \\
\longrightarrow & -\int_0^t \left(u_s(x)^2 - \int_{\mathbb{T}} u_s(y)^2 \mathrm{d}y \right) \\
& - \int_0^t \left(-2u_s(x) + \int_y u_s(y)\mathrm{d}y + 1 \right) \mathrm{d}s \\
& - \int_0^t \int_{\mathbb{T}} u_s^{\diamond 2}(y)\mathrm{d}y\mathrm{d}s,
\end{aligned}
$$

where

$$\int_0^t \int_{\mathbb{T}} u_s^{\diamond 2}(y)\mathrm{d}y\mathrm{d}s = \lim_{n\to\infty} \int_0^t \left(\int_{\mathbb{T}} u_s(\overline{\rho^n}(y))^2 \mathrm{d}y - \|\overline{\rho^n}\|_{L^2}^2 \right) \mathrm{d}s$$

denotes the integral of the renormalized square. The most singular contributions thus formally cancel each other. In fact it is not difficult to derive the following result:

Proposition 1 *Let u be an energy solution to Burgers equation and set for $\varphi \in C^\infty(\mathbb{T})$*

$$R_t^n(\varphi) := \int_0^t \int_{\mathbb{T}} \mathrm{d}x \varphi(x) w_s^n(x) \left(\mathrm{d}A_s(\Theta_x^n) - \left(u_s(\overline{\rho^n_x})^2 - \int_{\mathbb{T}} u_s(\overline{\rho^n}(y))^2 \mathrm{d}y \right) \mathrm{d}s - \frac{1}{12}\mathrm{d}s \right) \quad (5)$$

and

$$Q_t^n := \int_0^t \left(\int_{\mathbb{T}} u_s(\overline{\rho^n}(y))^2 \mathrm{d}y - \|\overline{\rho^n}\|_{L^2}^2 \right) \mathrm{d}s. \quad (6)$$

Assume that for all $\varphi \in C^\infty(\mathbb{T})$ the process $R^n(\varphi)$ converges to zero uniformly on compacts in probability, and that Q^n converges uniformly on compacts in probability to a process of zero quadratic variation. Assume furthermore that for all $T > 0$

$$\sup_{t\in[0,T], x\in\mathbb{T}} \mathbb{E}[e^{u_t(\Theta_x)}] < \infty,$$

where $\Theta_x := \lim_{n\to\infty} \Theta_x^n$. Then u is equal in distribution to $\partial_x \log w + \int_{\mathbb{T}} u_0(y)\mathrm{d}y$, where w is the unique-in-law martingale solution to the stochastic heat equation (2) with initial condition $w_0 = e^{u_0(\Theta_x)}$.

Proof This is a simple application of Ito's formula, see Appendix A of [GP18]. Actually we also need a uniform control of the p-variation of R^n for some $p > 2$ and we need Q^n to converge in q-variation for some $q < 2$ in order to control the integral $\int_0^t w_s^n(\varphi)\mathrm{d}Q_s^n$. Also the definition of R^n is not quite correct and $1/12$ should be replaced by $K_n \in \mathbb{R}$ with $K_n \to 1/12$. To simplify the presentation we ignore these subtleties here. □

Remark 1 The constant 1/12 that we subtract in the definition of R^n appears somewhat unexpectedly. But if we do not subtract it, then we will not be able to show that R^n converges to zero. This shows that the "FB solution h to the KPZ equation" (see [energy-uniqueness] for the definition) satisfies $h = \log w$, where w solves not (2) but the slightly modified equation

$$\partial_t w = \Delta w + w\left(\xi + \frac{1}{12}\right).$$

The constant 1/12 (which becomes 1/24 if we adapt the parameters in the equation to the popular choice $\partial_t h = \frac{1}{2}\Delta h + \frac{1}{2}|\partial_x h|^{\diamond 2} + \xi$) appears in many results about the KPZ equation.

We would like to apply this result to prove the uniqueness of energy solutions. The difficulty is that for a general energy solution we have no control of R^n and Q^n. We thus need more structure. This is the subject of the next section.

2.2 Forward-backward solutions

Recall that we have to show that R^n converges to zero, where

$$\begin{aligned}R^n_t(\varphi) &= \int_0^t \int_{\mathbb{T}} \mathrm{d}x \varphi(x) w^n_s(x) \left(\mathrm{d}A_s(\Theta^n_x) - \left(u_s\left(\overline{\rho^n_x}\right)^2 - \int_{\mathbb{T}} u_s\left(\overline{\rho^n}(y)\right)^2 \mathrm{d}y\right)\mathrm{d}s - \frac{1}{12}\mathrm{d}s\right)\\ &= \lim_{m\to\infty} \int_0^t \int_{\mathbb{T}} \mathrm{d}x \varphi(x) e^{u_s(\Theta^n_x)} \left(u_s\left(\overline{\rho^m}\right)^2\left(\partial_x \Theta^n_x\right) - \left(u_s\left(\overline{\rho^n_x}\right)^2 - \int_{\mathbb{T}} u_s\left(\overline{\rho^n}(y)\right)^2 \mathrm{d}y\right) - \frac{1}{12}\right)\mathrm{d}s\\ &=: \lim_{m\to\infty} R^{n,m}_t(\varphi).\end{aligned}$$

We also have to show that $Q^n_t = \int_0^t \left(\int_{\mathbb{T}} u_s\left(\overline{\rho^n}(y)\right)^2 \mathrm{d}y - \|\overline{\rho^n}\|^2_{L^2}\right)\mathrm{d}s$ converges. Both $R^{n,m}(\varphi)$ and Q^n are <u>additive functionals</u> of u, i.e. processes of the form

$$R^{n,m}_t(\varphi) = \int_0^t F^{n,m}(u_s)\mathrm{d}s, \qquad Q^n_t(\varphi) = \int_0^t G^n(u_s)\mathrm{d}s$$

for suitable $F^{n,m}, G^n : \mathcal{S}' \to \mathbb{R}$. We should therefore find a class of energy solutions for which we can control additive functionals. Note that since the space-time white noise has independent increments we would expect the solution u to Burgers equation to be a Markov process. So let us first see review some classical tools for controlling additive functionals of Markov processes:

Let X be a Markov process with values in a Polish state space E and denote its infinitesimal generator by $\mathcal{L}$. Let $F \in \mathrm{dom}(\mathcal{L})$ be such that also $F^2 \in \mathrm{dom}(\mathcal{L})$. Then by Dynkin's formula

$$M^F_t = F(X_t) - F(X_0) - \int_0^t \mathcal{L}F(X_s)\mathrm{d}s$$

defines a martingale with predictable quadratic variation

$$\langle M^F\rangle_t = \int_0^t (\mathcal{L}F^2 - 2F\mathcal{L}F)(X_s)\mathrm{d}s.$$

In particular, we can rewrite

$$\int_0^t (-\mathcal{L}F)(X_s)\mathrm{d}s = F(X_0) - F(X_t) + M^F_t, \tag{7}$$

and we can make two observations here: First, this representation gains regularity. Think for example of the generator of a diffusion, $\mathcal{L} = b(x)\partial_x + \frac{1}{2}\sigma^2(x)\partial_{xx}$. Then $(\mathcal{L}F^2 - 2F\mathcal{L}F)(x) = (\sigma(x)\partial_x F(x))^2$ involves only the first derivative of F and the boundary terms $F(X_0) - F(X_t)$ involve no derivatives of F, while $-\mathcal{L}F$ depends on the second derivative of F. By rewriting the additive functional in this way (assuming that our function is of the form $-\mathcal{L}F$) we thus got rid of one derivative! Note however that we achieve this by treating the finite variation additive functional as if it was a martingale, which in general only has finite p-variation for $p > 2$; in other words we gave up time regularity to gain space regularity.

Unfortunately this representation is not quite sufficient for us, because the "generator" of Burgers equation would also contain the term A, and given $G : \mathcal{S}' \to \mathbb{R}$ it seems difficult to solve the equation $-\mathcal{L}F = G$ (however this is to be interpreted in the first place). We therefore need a slightly more sophisticated argument: Assume now that X is a stationary Markov process with initial distribution μ, and that $\mathcal{L}$ is the infinitesimal generator of X on $L^2(\mu)$ (this is defined like the usual infinitesimal operator on $C_b(E)$, except that now the semigroup only has to be differentiated in $L^2(\mu)$). We write $\mathcal{L}^*$ for the $L^2(\mu)$-adjoint of X, and we note that for all $T > 0$ the process $(\hat{X}_t := X_{T-t})_{t\in[0,T]}$ is a stationary Markov process with generator $\mathcal{L}^*$. The easiest way to see this is to first show that the semigroup of $\hat{X}$ is the adjoint of the semigroup of X.

Assume now that $F \in \mathrm{dom}(\mathcal{L}) \cap \mathrm{dom}(\mathcal{L}^*)$ and also that $F^2 \in \mathrm{dom}(\mathcal{L}) \cap \mathrm{dom}(\mathcal{L}^*)$. Then (7) holds, and for the same reason also

$$\int_0^t (-\mathcal{L}^* F)(X_s)\mathrm{d}s = \int_{T-t}^T (-\mathcal{L}F)(\hat{X}_s)\mathrm{d}s = F(\hat{X}_{T-t}) - F(\hat{X}_T) + \hat{M}^F_T - \hat{M}^F_{T-t}, \qquad (8)$$

where $\hat{M}^F$ is a martingale in the filtration generated by $\hat{X}$ (the backward filtration of X) with predictable quadratic variation $\langle M^F\rangle_t = \int_0^t (\mathcal{L}^* F^2 - 2F\mathcal{L}^* F)(X_s)\mathrm{d}s$. Since $F(\hat{X}_{T-t}) = F(X_t)$ and $F(\hat{X}_T) = F(X_0)$, we can add (7) and (8) and obtain

$$\int_0^t (-2\mathcal{L}_S F)(X_s)\mathrm{d}s = \int_0^t (-\mathcal{L}^* F - \mathcal{L}F)(X_s)\mathrm{d}s = M^F_t + \hat{M}^F_T - \hat{M}^F_{T-t},$$

where $\mathcal{L}_S := \frac{1}{2}(\mathcal{L} + \mathcal{L}^*)$ is the symmetric part of $\mathcal{L}$. This new representation of our additive functional has two main advantages compared to the previous one: It does not involve any more boundary terms, and now we need to solve the equation $-\mathcal{L}_S F = G$, and as we will see (or rather postulate) below, the symmetric part of the generator does not involve the process A. Moreover, we got rid of the boundary terms and can now directly apply martingale inequalities to our additive functional.

We cannot directly apply these arguments for energy solutions of the stochastic Burgers equation, because a priori do not know if the energy solution is a Markov process, and we have no idea about its infinitesimal generator. So we need to find a new notion of solution, where we can proceed as above. This definition was found by [GJ13]:

Definition 3 *A stationary energy solution to (4) is called a* <u>*forward-backward solution*</u> *(also* <u>*FB-solution*</u>*) if additionally for all times $T > 0$ also the process $\hat{u}_t := u_{T-t}$ is an energy solution to*

$$\partial_t \hat{u} = \Delta \hat{u} - \partial_x \hat{u}^2 + \sqrt{2}\partial_x \hat{\xi},$$

where $\hat{\xi}$ is a space-time white noise that is adapted to the backward filtration (the one generated by $\hat{u}$), and with the obvious adaptation of the definition to this equation with opposite sign of the nonlinearity. In that case we get for the nonlinearity $\hat{A}$ of $\hat{u}$ that $\hat{A}_t = -(A_T - A_{T-t})$.

Remark 2 In the literature FB-solutions are also called energy solutions. I use a separate name here for pedagogical reasons.

Intuitively, this definition says that the Burgers nonlinearity is antisymmetric in $L^2(\mu)$, where μ is the law of the white noise, and the linear part of the equation is symmetric. By the considerations above this suggests that we should be able to control additive functionals of FB-solutions in terms of the generator of the linear dynamics, which we understand much better than the nonlinear part!

In the following result we will need the notion of a cylinder function, which is just a function $F : \mathcal{S}' \to \mathbb{R}$ of the form $F(u) = f(u(\varphi_1), \ldots, u(\varphi_m))$, where $f \in C^2(\mathbb{R}^m, \mathbb{R})$ is such that $\partial^\alpha f$ grows at most polynomially whenever $|\alpha| \leqslant 2$, and where $\varphi_1, \ldots, \varphi_m \in C^\infty(\mathbb{T})$.

Lemma 1 (Itô trick) *Let u be a FB-solution to Burgers equation. Let F be a cylinder function and define*

$$\mathcal{L}_S F(u) := \sum_{i=1}^m \partial_i f(u(\varphi_1), \ldots, u(\varphi_n)) u(\Delta \varphi_i) + \sum_{i,j=1}^m \partial_{ij} f(u(\varphi_1), \ldots, u(\varphi_n)) \langle \partial_x \varphi_i, \partial_x \varphi_j \rangle_{L^2}$$

and

$$\mathcal{E}(F(u)) := 2 \int \left| \sum_{i=1}^n \partial_i f(u(\varphi_1), \ldots, u(\varphi_n)) \partial_x \varphi_i(x) \right|^2 \mathrm{d}x.$$

Then we have for all $p \geqslant 1$

$$\mathbb{E}\left[\sup_{t \leqslant T} \left| \int_0^t \mathcal{L}_S F(u_s) \mathrm{d}s \right|^p \right] \lesssim T^{p/2} \mathbb{E}[\mathcal{E}(F(u_0))^{p/2}].$$

Proof We apply Ito's formula to $F(u_t)$ and obtain

$$F(u_t) = F(u_0) + \int_0^t \mathcal{L}_S F(u_s) \mathrm{d}s + \int_0^t \sum_{i=1}^m \partial_i f(u_s(\varphi_1), \ldots, u_s(\varphi_n)) \mathrm{d}A_s(\varphi_i) + M_t^F,$$

where M^F is a continuous martingale with quadratic variation

$$\langle M^F \rangle_t = 2 \int_0^t \sum_{i,j=1}^m \partial_i f(u_s(\varphi_1), \ldots, u_s(\varphi_n)) \partial_j f(u_s(\varphi_1), \ldots, u_s(\varphi_n)) \langle \partial_x \varphi_i, \partial_x \varphi_j \rangle \mathrm{d}s = \int_0^t \mathcal{E} F(u_s) \mathrm{d}s.$$

Similarly, we have

$$\begin{aligned} F(\hat{u}_T) &= F(\hat{u}_{T-t}) + \int_{T-t}^T \mathcal{L}_S F(\hat{u}_s) \mathrm{d}s + \int_{T-t}^T \sum_{i=1}^m \partial_i f(\hat{u}_s(\varphi_1), \ldots, \hat{u}_s(\varphi_n)) \mathrm{d}\hat{A}_s(\varphi_i) + \hat{M}_T^F - \hat{M}_{T-t}^F \\ &= F(u_t) + \int_0^t \mathcal{L}_S F(u_s) \mathrm{d}s - \int_0^t \sum_{i=1}^m \partial_i f(u_s(\varphi_1), \ldots, u_s(\varphi_n)) \mathrm{d}A_s(\varphi_i) + \hat{M}_T^F - \hat{M}_{T-t}^F, \end{aligned}$$

and $\langle \hat{M}^F \rangle_t = \int_0^t \mathcal{E} F(\hat{u}_s) \mathrm{d}s$. Thus

$$\int_0^t \mathcal{L}_S F(u_s) \mathrm{d}s = -\frac{1}{2}(M_t^F + \hat{M}_T^F - \hat{M}_{T-t}^F),$$

and the claimed bound follows from the Burkholder-Davis-Gundy inequality together with the stationarity of u. $\square$

To apply this result, we must be able to solve the Poisson equation $\mathcal{L}_S F = G$. This is actually not as hopeless as it may seem, since $\mathcal{L}_S$ allows for a simple description when we combine it with the *Wiener-Ito chaos decomposition* in $L^2(\mu)$. To introduce the chaos decomposition, let us first define the Hermite polynomials

$$H_n(x) := e^{x^2/2}(-1)^n \partial_x^n e^{-x^2/2}.$$

By induction we see that H_n is a degree n polynomial with term of leading order x^n. If X is a standard normal variable, then $H_n(X)$ and $H_m(X)$ are orthogonal for $n \neq m$, and

$$\begin{aligned}\mathbb{E}[H_n(X)^2] &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} H_n(x)(-1)^n \partial_x^n e^{-x^2/2} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \partial_x^n H_n(x) e^{-x^2/2} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} n! e^{-x^2/2} \mathrm{d}x = n!,\end{aligned}$$

where we used that H_n is a degree n polynomial with leading coefficient 1.

Let η be a white noise on $\mathbb{T}$. For $\varphi \in C^\infty(\mathbb{T})$ with $\varphi \neq 0$ and $n \geqslant 1$ we define

$$W_n(\varphi^{\otimes n})(\eta) := \|\varphi\|_{L^2}^n H_n\left(\frac{\eta(\varphi)}{\|\varphi\|_{L^2}}\right),$$

and $W_n(0^{\otimes n}) := 0$, which satisfies

$$\mathbb{E}[|W_n(\varphi^{\otimes n})|^2] = n! \|\varphi\|_{L^2}^{2n} = n! \|\varphi^{\otimes n}\|_{L^2}^2,$$

and by polarization also $\mathbb{E}[W_n(\varphi^{\otimes n}) W_n(\psi^{\otimes n})] = n! \langle \varphi^{\otimes n}, \psi^{\otimes n} \rangle_{L^2}$. Through the explicit multilinear polarization formula

$$W_n(\varphi_1 \otimes \ldots \otimes \varphi_n) := \frac{1}{2^n n!} \sum_{\varepsilon_1, \ldots, \varepsilon_n = \pm 1} \varepsilon_1 \ldots \varepsilon_n W_n((\varepsilon_1 \varphi_1 + \cdots + \varepsilon_n \varphi_n)^{\otimes n}).$$

we obtain a unique extension of the map W_n from functions of the form $\varphi^{\otimes n}$ to $\varphi_1 \otimes \ldots \otimes \varphi_n$ as a symmetric multilinear form, and then we still have

$$\begin{aligned}&\mathbb{E}[|W_n(\varphi_1 \otimes \ldots \otimes \varphi_n)|^2] \\ &\quad = \frac{1}{2^{2n}(n!)^2} \sum_{\varepsilon_1, \varepsilon_1', \ldots, \varepsilon_n, \varepsilon_n' = \pm 1} \varepsilon_1 \ldots \varepsilon_n \varepsilon_1' \ldots \varepsilon_n' n! \langle (\varepsilon_1 \varphi_1 + \cdots + \varepsilon_n \varphi_n)^{\otimes n}, (\varepsilon_1' \varphi_1 + \cdots + \varepsilon_n' \varphi_n)^{\otimes n} \rangle_{L^2} \\ &\quad = n! \|\varphi_1 \widetilde{\otimes \ldots \otimes} \varphi_n\|_{L^2}^2 =: n! \|\varphi_1 \otimes \ldots \otimes \varphi_n\|_{L_s^2}^2,\end{aligned}$$

where

$$\varphi_1 \widetilde{\otimes \ldots \otimes} \varphi_n(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varphi_1(x_{\sigma(1)}) \ldots \varphi_n(x_{\sigma(n)})$$

is the symmetrization of $\varphi_1 \otimes \ldots \otimes \varphi_n$, with the symmetric group $\mathfrak{S}_n$. By the triangle inequality we have

$$\|\varphi_1 \otimes \ldots \otimes \varphi_n\|_{L_s^2} = \|\varphi_1 \widetilde{\otimes \ldots \otimes} \varphi_n\|_{L^2} \leqslant \|\varphi_1 \otimes \ldots \otimes \varphi_n\|_{L^2}.$$

By linearity we can extend W_n as a linear map on linear combinations of such functions, which are dense in L^2. The isometry property above thus allows us to extend W_n to all of $L^2(\mathbb{T}^n)$, and we have

$$\mathbb{E}[W_n(\varphi_n)W_m(\varphi_m)] = \delta_{m,n} m! \langle \varphi_n, \varphi_m \rangle_{L^2_s(\mathbb{T}_m)}, \qquad \varphi_n \in L^2_s(\mathbb{T}^n), \varphi_m \in L^2_s(\mathbb{T}^m),$$

where

$$\langle \varphi_n, \varphi_m \rangle_{L^2_s(\mathbb{T}_m)} := \langle \tilde{\varphi}_n, \tilde{\varphi}_m \rangle_{L^2}$$

for the symmetrizations $\tilde{\varphi}_n$ and $\tilde{\varphi}_m$ of φ_n and φ_m. The space $\{W_n(\varphi_n) : \varphi_n \in L^2_s(\mathbb{T}_m)\}$ is called the n-th homogeneous chaos, and by construction it is a closed subspace of $L^2(\mu)$. One can show that any $F \in L^2(\mu)$ has a representation, the so called chaos expansion, of the form

$$F = \sum_{n=0}^{\infty} W_n(\varphi_n)$$

with $\varphi_n \in L^2(\mathbb{T}^n)$, where $W_0(\varphi_0) := \varphi_0$, and that

$$\|F\|^2_{L^2(\mu)} = \sum_{n=0}^{\infty} n! \|\varphi_n\|^2_{L^2_s(\mathbb{T}^n)}.$$

Indeed, the space $\mathcal{H} \subset L^2(\mu)$ of all functions with a chaos expansion is closed, and it contains all polynomials in variables $\eta(\varphi_1), \dots, \eta(\varphi_m)$ with $m \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_m \in C^\infty(\mathbb{T})$. By Stone-Weierstraß it also contains all compactly supported continuous functions in these variables, and then it follows from the monotone class theorem that any F that is orthogonal to $\mathcal{H}$ must be zero.

Moreover, if we identify functions $f, g \in L^2(\mathbb{T}^n)$ whenever $\tilde{f} = \tilde{g}$, then the equivalence class of the kernel φ_n in the chaos expansion is unique.

Lemma 2 *The operator $\mathcal{L}_S$ is closable in $L^2(\mu)$. We denote the closure also by $\overline{\mathcal{L}_S}$, and its domain is*

$$\operatorname{dom}\left(\overline{\mathcal{L}_S}\right) = \left\{ F = \sum_{n=0}^{\infty} W_n(\varphi_n) : \sum_{n=0}^{\infty} n! \|\Delta \varphi_n\|^2_{L^2} < \infty \right\},$$

where $\Delta \varphi_n := \sum_{i=1}^n \partial_{ii} \varphi_n$ and the derivatives are taken in the weak sense. For $F = \sum_{n=0}^\infty W_n(\varphi_n) \in \operatorname{dom}\left(\overline{\mathcal{L}_S}\right)$ we have

$$\overline{\mathcal{L}_S} F = \sum_{n=0}^{\infty} W_n(\Delta \varphi_n).$$

Moreover,

$$\mathbb{E}[\mathcal{E}(W_n(\varphi_n))] = 2n! \sum_{i=1}^{n} \|\partial_i \varphi_n\|^2_{L^2}.$$

Proof It is easy to check that $\overline{\mathcal{L}_S}$ is a closed operator in $L^2(\mu)$, essentially because Δ is a closed operator in $L^2(\mathbb{T}^n)$ with domain $H^2(\mathbb{T}^n)$, the L^2 Sobolev space of twice differentiable functions. So we only have to show that $\overline{\mathcal{L}_S}$ extends $\mathcal{L}_S$. We first show that for $\varphi \in C^\infty(\mathbb{T})$ with $\|\varphi\|_{L^2} = 1$ we have

$$\begin{aligned}\mathcal{L}_S W_n(\varphi^{\otimes n}) &= \mathcal{L}_S H_n(\eta(\varphi)) = H_n'(\eta(\varphi))\eta(\Delta\varphi) + H_n''(\eta(\varphi))\|\partial_x \varphi\|^2_{L^2} \\ &= H_n'(\eta(\varphi))\eta(\Delta\varphi) - H_n''(\eta(\varphi))\langle \varphi, \Delta\varphi \rangle_{L^2}.\end{aligned}$$

Below we will show that $H_n' = nH_{n-1}$ and thus also $H_n'' = n(n-1)H_{n-2}$, and also that

$$\begin{aligned} nH_{n-1}(\eta(\varphi))\eta(\Delta\varphi) &= nW_n(\varphi^{\otimes n-1} \otimes \Delta\varphi) + n(n-1)W_{n-2}(\varphi^{\otimes n-2})\langle\varphi, \Delta\varphi\rangle_{L^2} \\ &= W_n(\Delta\varphi^{\otimes n}) + n(n-1)H_{n-2}(\eta(\varphi))\langle\varphi, \Delta\varphi\rangle_{L^2}. \end{aligned}$$

Therefore, $\mathcal{L}_S W_n(\varphi^{\otimes n}) = W_n(\Delta\varphi^{\otimes n})$. But as we saw above, linear combinations (in n and φ) of $H_n(\eta(\varphi))$ with $\|\varphi\|_{L^2} = 1$ are dense in $L^2(\mu)$, and therefore we obtain for all cylinder functions F with chaos expansion $F = \sum_n W_n(\varphi_n)$ that $\mathcal{L}_S$ acts as $\mathcal{L}_S F = \sum_n W_n(\Delta\varphi_n)$.

Consider now $\mathcal{E}(H_n(\eta(\varphi)))$ for $\varphi \in C^\infty(\mathbb{T})$ with $\|\varphi\|_{L^2} = 1$:

$$\begin{aligned} \mathbb{E}[\mathcal{E}(H_n(\eta(\varphi)))] &= 2\int \mathbb{E}[|H_n'(\eta(\varphi))\partial_x\varphi(x)|^2]\mathrm{d}x \\ &= 2\int \mathbb{E}[|nH_{n-1}(\eta(\varphi))\partial_x\varphi(x)|^2]\mathrm{d}x \\ &= 2n!n\|\partial_x\varphi\|_{L^2}^2 \\ &= 2n!\sum_{i=1}^n \|\partial_i\varphi^{\otimes n}\|_{L^2}^2. \end{aligned}$$

Since W_n is a linear continuous functional, we get by polarization, linearity and continuity:

$$\mathbb{E}[\mathcal{E}(W_n(\varphi_n))] = 2n!\sum_{i=1}^n \|\partial_i\varphi_n\|_{L^2}^2$$

for all symmetric $\varphi_n \in L^2(\mathbb{T}^n)$. □

In the following we omit the line above $\overline{\mathcal{L}_S}$ and simply write $\mathcal{L}_S$ and $\mathrm{dom}(\mathcal{L}_S)$. In the proof we used the following auxiliary result:

Lemma 3 *Chaos expansion and Hermite polynomials have the following properties:*

$$H_n'(x) = nH_{n-1}(x)$$

and

$$W_n(\varphi^{\otimes n})W_1(\psi) = W_{n+1}(\varphi^{\otimes n} \otimes \psi) + nW_{n-2}(\varphi^{\otimes n-2})\langle\varphi, \psi\rangle_{L^2}.$$

Proof Note that for $k \geqslant 0$ we have

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} H_n'(x)H_k(x)e^{-x^2/2}\mathrm{d}x &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \partial_x H_n(x)(-1)^k\partial_x^k(e^{-x^2/2})\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \partial_x^{k+1}H_n(x)e^{-x^2/2}\mathrm{d}x = \delta_{k,n-1}n!, \end{aligned}$$

and since $(k!)^{-1/2}H_k$ is an orthonormal basis in $L^2((2\pi)^{-1/2}e^{-x^2/2})$ we get $H_n' = nH_{n-1}$. For the second statement see [Nua06, Proposition 1.1.2]. □

As an application, let us look at the Burgers drift. Recall that our definition of martingale solutions depends on a specific mollifier that is used to construct the nonlinearity. But for FB-solutions the specific mollifier does in fact not matter, and any mollifier will lead to the same limit: Let $\delta \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \delta(x)\mathrm{d}x = 1$ and consider

$$\int_0^t \int_{\mathbb{T}} \partial_x u_s \left(\overline{\delta_x^n}\right)^2 \varphi(x)\mathrm{d}x\mathrm{d}s.$$

Our aim is to show that this converges to $A_t(\varphi)$. For that purpose note that

$$
\begin{aligned}
\int_{\mathbb{T}} \partial_x u_s \left(\overline{\delta^n_x}\right)^2 \varphi(x)\mathrm{d}x &= \int_{\mathbb{T}} \partial_x \left(\left\|\overline{\delta^n_x}\right\|_{L^2}^2 H_2\left(\frac{u_s\left(\overline{\delta^n_x}\right)}{\left\|\overline{\delta^n_x}\right\|_{L^2}}\right) + \int_{\mathbb{T}} \left|\sum_k \delta^n(x+k-y)\right|^2 \mathrm{d}y \right) \varphi(x)\mathrm{d}x \\
&= \int_{\mathbb{T}} \partial_x \left(\left\|\overline{\delta^n_x}\right\|_{L^2}^2 H_2\left(\frac{u_s\left(\overline{\delta^n_x}\right)}{\left\|\overline{\delta^n_x}\right\|_{L^2}}\right) + \int_{\mathbb{T}} \left|\sum_k \delta^n(k-y)\right|^2 \mathrm{d}y \right) \varphi(x)\mathrm{d}x \\
&= \int_{\mathbb{T}} \partial_x \left(\left\|\overline{\delta^n_x}\right\|_{L^2}^2 H_2\left(\frac{u_s\left(\overline{\delta^n_x}\right)}{\left\|\overline{\delta^n_x}\right\|_{L^2}}\right)\right) \varphi(x)\mathrm{d}x \\
&= -\int_{\mathbb{T}} \left\|\overline{\delta^n_x}\right\|_{L^2}^2 H_2\left(\frac{u_s\left(\overline{\delta^n_x}\right)}{\left\|\overline{\delta^n_x}\right\|_{L^2}}\right) \partial_x\varphi(x)\mathrm{d}x \\
&= -\int_{\mathbb{T}} W_2\left(\overline{\delta^n_x}^{\otimes 2}\right)(u_s)\partial_x\varphi(x)\mathrm{d}x \\
&= W_2\left(-\int_{\mathbb{T}} \overline{\delta^n_x}^{\otimes 2}\partial_x\varphi(x)\mathrm{d}x\right)(u_s),
\end{aligned}
$$

where in the last step we used that W_2 is linear and continuous in $L^2(\mathbb{T}^2)$ by construction. Combining the representation with the Itô trick and the representation of the generator that we found before, it is not difficult to derive the following result:

Lemma 4 (Burgers nonlinearity) *Let u be a FB-solution to Burgers equation. The process A satisfies for all $\delta \in C_c^\infty$ with $\int_{\mathbb{R}} \delta(x)\mathrm{d}x = 1$, for all $T > 0$ and all $\varphi \in C^\infty(\mathbb{T})$*

$$
\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t \int_{\mathbb{T}} \partial_x u_s \left(\overline{\delta^n_x}\right)^2 \varphi(x)\mathrm{d}x - A_t(\varphi)\right|^2\right] = 0.
$$

Proof (Sketch of proof)

By definition of A we have

$$
\begin{aligned}
\int_0^t \int_{\mathbb{T}} \partial_x u_s \left(\overline{\delta^n_x}\right)^2 \varphi(x)\mathrm{d}x - A_t(\varphi)A_t(\varphi) &= \int_0^t \int_{\mathbb{T}} \partial_x u_s \left(\overline{\delta^n_x}\right)^2 \varphi(x)\mathrm{d}x + \lim_{m\to\infty}\int_0^t (u_s * \rho^m)^2(\partial_x\varphi)\mathrm{d}s \\
&= \lim_{m\to\infty}\int_0^t W_2\left(-\int_{\mathbb{T}}\left(\overline{\delta^n_x}^{\otimes 2} - \overline{\rho^m_x}^{\otimes 2}\right)\partial_x\varphi(x)\mathrm{d}x\right)(u_s)\mathrm{d}s.
\end{aligned}
$$

To apply the Itô trick, we see from Lemma 2 that we need to solve

$$
\Delta\Phi^{m,n}(y_1,y_2) = -\int_{\mathbb{T}}\left(\overline{\delta^n_x}(y_1,y_2) - \overline{\rho^m_x}(y_1,y_2)\right)\partial_x\varphi(x)\mathrm{d}x =: \Psi^{m,n}(y_1,y_2).
$$

Since the right hand side satisfies

$$
\mathcal{F}\Psi^{m,n}(0,0) = \int_{\mathbb{T}^2}\Psi^{m,n}(y_1,y_2)\mathrm{d}y_1\mathrm{d}y_2 = 0,
$$

this equation is easy to solve with the help of Fourier coordinates: simply set

$$
\mathcal{F}\Phi^{m,n}(k_1,k_2) = -\frac{\mathcal{F}\Psi^{m,n}(k_1,k_2)}{|2\pi k_1|^2 + |2\pi k_2|^2}.
$$

The Itô trick then gives

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t W_2(\Psi^{m,n})(u_s)\mathrm{d}s\right|^2\right] &\lesssim T\mathbb{E}[\mathcal{E}(W_2(\Phi^{m,n}))(u_0)]\\ &\simeq \sum_{i=1}^{2}\|\partial_i\Phi^{m,n}\|_{L^2}^2\\ &= \sum_{k\in\mathbb{Z}^2}(|2\pi k_1|^2+|2\pi k_2|^2)|\mathcal{F}\Phi^{m,n}(k_1,k_2)|^2\\ &= \sum_{k\in\mathbb{Z}^2}\frac{|\mathcal{F}\Psi^{m,n}(k_1,k_2)|^2}{|2\pi k_1|^2+|2\pi k_2|^2}.\end{aligned}$$

To go further, we need to compute $\mathcal{F}\Psi^{m,n}(k_1,k_2)$:

$$\begin{aligned}\mathcal{F}\Psi^{m,n}(k_1,k_2) &= -\int_{\mathbb{T}^2} e^{-2\pi i(k_1y_1+k_2y_2)}\int_{\mathbb{T}}\left(\overline{\delta^n_x}(y_1)\overline{\delta^n_x}(y_2)-\overline{\rho^m_x}(y_1)\overline{\rho^m_x}(y_2)\right)\partial_x\varphi(x)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2\\ &= -\int_{\mathbb{T}}\left(\mathcal{F}\overline{\delta^n_x}(k_1)\mathcal{F}\overline{\delta^n_x}(k_2)-\mathcal{F}\overline{\rho^m_x}(k_1)\mathcal{F}\overline{\rho^m_x}(k_2)\right)\partial_x\varphi(x)\mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}\overline{\delta^n_x}(k_1) &= \int_{\mathbb{T}} e^{-2\pi i k_1 y}\sum_{\ell}\delta^n(x-y+\ell)\mathrm{d}y\\ &= \sum_{\ell}\int_{\mathbb{T}} e^{-2\pi i k_1(y-\ell)}\delta^n(x-(y-\ell))\mathrm{d}y\\ &= \int_{\mathbb{R}} e^{-2\pi i k_1 y} n\delta(n(x-y))\mathrm{d}y\\ &= e^{-2\pi i k_1 x}\mathcal{F}_{\mathbb{R}}\delta\left(-\frac{k_1}{n}\right),\end{aligned}$$

where $\mathcal{F}_{\mathbb{R}}$ is the Fourier transform on $\mathbb{R}$. Thus

$$\mathcal{F}\Psi^{m,n}(k_1,k_2) = \left(\mathcal{F}_{\mathbb{R}}\rho\left(\frac{-k_1}{n}\right)\mathcal{F}_{\mathbb{R}}\rho\left(\frac{-k_2}{n}\right)-\mathcal{F}_{\mathbb{R}}\delta\left(\frac{-k_1}{n}\right)\mathcal{F}_{\mathbb{R}}\delta\left(\frac{-k_2}{n}\right)\right)\mathcal{F}(\partial_x\varphi)(k_1+k_2).$$

From here the claim follows after a lengthy but elementary computation: subtract and add 1 inside the bracket to gain a factor $(|k_1|+|k_2|)^\alpha n^{-\alpha}$ for $\alpha\in(0,1)$ and use that $\mathcal{F}(\partial_x\varphi)(k_1+k_2)$ decays faster than polynomially in (k_1+k_2). We need this decay to make the series finite, because $\sum_{k\in\mathbb{Z}^2\setminus\{0\}}(|2\pi k_1|^2+|2\pi k_2|^2)^{-1}=\infty$. See [GP18, Proposition 3.15] for details. □

Recall that our aim is to prove the convergence of the additive functionals

$$R^{n,m}_t = \int_0^t\int_{\mathbb{T}}\mathrm{d}x\varphi(x)e^{u_s(\Theta^n_x)}\left(u_s\left(\overline{\rho^m}\right)^2(\partial_x\Theta^n_x)-\left(u_s\left(\overline{\rho^n_x}\right)^2-\int_{\mathbb{T}}u_s\left(\overline{\rho^n}(y)\right)^2\mathrm{d}y\right)-\frac{1}{12}\right)\mathrm{d}s$$

and

$$Q^n_t = \int_0^t\left(\int_{\mathbb{T}}u_s\left(\overline{\rho^n}(y)\right)^2\mathrm{d}y-\|\overline{\rho^n}\|^2_{L^2}\right)\mathrm{d}s.$$

The process Q^n looks very much like the Burgers drift and can be controlled by similar arguments. However, the integrand for $R^{n,m}$ does not have a finite chaos expansion because it contains $e^{u_s(\Theta^n_x)}$. By working with the Wick exponential it should still possible to derive the explicit chaos expansion of the integrand and to control $R^{n,m}$ using similar arguments as for the Burgers drift. But we prefer to take another, less explicit and maybe less computationally intense approach.

Note that in the proof of Lemma 4 we first solved the Poisson equation $\mathcal{L}_S\Phi^{m,n} = \Psi^{m,n}$, but the bound we derived using the Itô trick only involved $\mathcal{F}\Psi^{m,n}$ and not the solution $\Phi^{m,n}$ itself. This can be generalized as follows: We define for $F \in L^2(\mu)$ the "$\mathcal{H}^1$ norm" (strictly speaking this is only a seminorm on the subspace of L^2 where it is finite)

$$\|F\|_1^2 := \mathbb{E}[\mathcal{E}(F)(\eta)] \in [0,\infty]$$

and by duality the "$\mathcal{H}^{-1}$ norm"

$$\|F\|_{-1} = \sup_{G:\|G\|_1\leqslant 1} \mathbb{E}[F(\eta)G(\eta)] \in [0,\infty].$$

Note that $\|F\|_{-1} = \infty$ whenever $\mathbb{E}[F(\eta)] \neq 0$ because constant functions G satisfy $\|G\|_1 = 0$ (because then $\mathcal{L}_S G = 0$). With the help of the $\mathcal{H}^{-1}$ norm we can estimate the additive functional $\int_0^\cdot F(u_s)\mathrm{d}s$ without first solving the Poisson equation:

Lemma 5 (Kipnis-Varadhan inequality) *Let $F \in L^2(\mu)$ be such that $\|F\|_{-1} < \infty$. Then*

$$\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t F(u_s)\mathrm{d}s\right|^2\right] \lesssim T\|F\|_{-1}^2.$$

Proof Since $\mathcal{L}_S$ is the generator of a contraction semigroup $(T^S_t)_{t\geqslant 0}$ on $L^2(\mu)$ (the semigroup associated to the linear SPDE $\partial_t X = \Delta X + \sqrt{2}\partial_x\xi$), we can solve the resolvent equation for $\lambda > 0$ and any right hand side:

$$(\lambda - \mathcal{L}_S)G_\lambda = F \qquad \Leftrightarrow \qquad G_\lambda = \int_0^\infty e^{-\lambda t} T^S_t F \mathrm{d}t.$$

Indeed, at least formally we get from an integration by parts

$$-\mathcal{L}_S G_\lambda = -\int_0^\infty e^{-\lambda t}\partial_t T^S_t F\mathrm{d}t = F + \int_0^\infty (-\lambda)e^{-\lambda t}T^S_t F\mathrm{d}t = F - \lambda G_\lambda.$$

We want to apply to the Itô trick to $-\mathcal{L}_S G_\lambda$, which requires a bound on $\|G_\lambda\|_1^2$. But

$$\begin{aligned}\|G_\lambda\|_1^2 &\simeq \mathbb{E}[G_\lambda(\eta)(-\mathcal{L}_S)G_\lambda(\eta)] \\ &= \mathbb{E}[G_\lambda(\eta)(-\mathcal{L}_S)G_\lambda(\eta)] \\ &= \mathbb{E}[G_\lambda(\eta)F(\eta)] - \lambda\mathbb{E}[|G_\lambda(\eta)|^2] \\ &\leqslant \|F\|_{-1}\|G_\lambda\|_1 - \lambda\|G_\lambda\|^2_{L^2(\mu)},\end{aligned}$$

and therefore $\|G_\lambda\|_1 \leqslant \|F\|_{-1}$. Plugging this into the inequality above, we also get

$$\lambda\|G_\lambda\|^2_{L^2(\mu)} \leqslant \|F\|^2_{-1}.$$

Therefore, applying the Itô trick and using that u_s is for all times a white noise,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t F(u_s)\mathrm{d}s\right|^2\right] &\lesssim \mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t \lambda G_\lambda(u_s)\mathrm{d}s\right|^2\right] + \mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t -\mathcal{L}_S G_\lambda(u_s)\mathrm{d}s\right|^2\right] \\ &\lesssim T^2\mathbb{E}[|\lambda G_\lambda(u_0)|^2] + T\|G_\lambda\|_1^2 \\ &\leqslant T^2\lambda^2\|G_\lambda\|^2_{L^2(\mu)} + T\|F\|^2_{-1} \\ &\leqslant (T^2\lambda + T)\|F\|^2_{-1}.\end{aligned}$$

As $\lambda > 0$ was arbitrary, it now suffices to send $\lambda \to 0$. □

To apply this result we need to test the integrand of $R^{n,m}$ against a test function and to control the expectation in terms of the $\mathcal{H}^1$ norm of the test function. We can do this by representing the $\mathcal{H}^1$ norm in terms of the Malliavin derivative (if you are familiar with the Malliavin derivative have a look at the definition of $\mathcal{E}$ and rewrite it in terms of the Malliavin derivative) and by applying integration by parts with respect to the white noise measure. Like that we finally obtain the following result:

Theorem 1 *Let u be a FB-solution. Then the conditions of Proposition 1 are satisfied and in particular the distribution of u is unique.*

This is carried out in [GP18], here we do not go further into detail and instead look at an application of energy solutions.

3 Application: scaling limits

One reason why KPZ and Burgers equation are so popular is that they arise as universal scaling limits of $(1+1)$–dimensional interface growth models (one time dimension, one space dimension) in the so called *weakly asymmetric regime*. This is known as the *weak KPZ universality conjecture*, although by now the convergence has been established for so many models that it is more of a result than a conjecture. As the name suggests, there also is a *(strong) KPZ universality conjecture*, which claims that all $(1+1)$–dimensional interface growth models with some qualitative assumptions show the same large scale behavior as the KPZ equation. This is a very active field of research and there are many indications that the conjecture is true, although rigorous proofs exist so far almost exclusively for the one-dimensional marginal distributions (like a central limit theorem at fixed times rather than Donsker's invariance principle for the entire process), and they mostly rely on good algebraic properties of the models (they work for "stochastically integrable" models).

Energy solutions / FB solutions are a good tool for proving the convergence of Markovian interface growth models to Burgers equation (and also to the KPZ equation) because their martingale description is very reminiscent of the martingale problem description of a Markov process. However, since FB-solutions also require stationarity and an explicit decomposition of the generator into symmetric and antisymmetric part, we need to know the invariant measure of our Markov process explicitly in order to apply them. Luckily there are many interesting models for which this is the case, see e.g. [GJ14, GJS15, DGP17, GP16]. Here I will focus on the relatively simple weakly asymmetric simple exclusion process, for which the convergence was already shown by Bertini and Giacomin [BG97] who implemented a Cole-Hopf transform for this process. Instead of following their approach, we will use FB-solutions to prove the convergence.

Let $n \in \mathbb{N}$ and set $\mathbb{Z}_n = \mathbb{Z}/(n\mathbb{Z})$ and define $p_n : \mathbb{Z}_n \to [0,1]$ by

$$p_n(k) = \begin{cases} \frac{1}{2}(1 - n^{-1/2}), & k = -1, \\ \frac{1}{2}(1 + n^{-1/2}), & k = 1, \\ 0, & \text{else}. \end{cases}$$

So p_n is the jump probability of a weakly asymmetric random walk on $\mathbb{Z}_n$, and setting

$$q_n(k, \ell) = \begin{cases} p_n(\ell - k), & \ell \neq k, \\ -\sum_m p_n(m) & \ell = k, \end{cases}$$

we obtain the transition rates of a continuous time random walk. Note that $-\sum_m p_n(m) = -1$. We consider independent particles on $\mathbb{Z}_n$ that all follow the continuous time random walk with transition rates q_n, but that are not allowed to hop on top of each other (there is an *exclusion rule*). The resulting process is called the *weakly asymmetric simple exclusion process (WASEP)* and if we do not care about the individual particles but only about their joint configuration, then the state space of the process can be chosen as $S_n = \{0,1\}^{\mathbb{Z}_n}$, where for $\eta \in S_n$ we have $\eta(k) = 1$ if there is a particle at site k, and $\eta(k) = 0$ otherwise. Note that S_n is finite, so we are dealing with a continuous time Markov chain on a finite state space. Its infinitesimal generator or Q-matrix acts on $f : S_n \to \mathbb{R}$ as follows:

$$\mathcal{L}^n f(\eta) = \sum_{k,\ell \in \mathbb{Z}_n} \eta(k)(1 - \eta(\ell)) p_n(\ell - k)(f(\eta^{k,\ell}) - f(\eta)),$$

where

$$\eta^{k,\ell}(m) = \begin{cases} \eta(\ell), & m = k, \\ \eta(k), & m = \ell, \\ \eta(m), & \text{else}, \end{cases}$$

is the configuration with the occupation status at sites k and ℓ exchanged. The factor $\eta(k)(1 - \eta(\ell))p_n(\ell - k)$ in the generator is equal to $p_n(\ell - k)$ exactly if there is a particle in k and no particle in ℓ, so only in that case the particle in k is trying to move to the empty site ℓ with rate $p_n(\ell - k)$. Otherwise this factor is 0 and nothing changes.

One can show that for all $\rho \in [0,1]$ the product Bernoulli measure

$$\mu^n_\rho(\{\eta\}) = \rho^{\sum_{k \in \mathbb{Z}_n} \eta(k)} (1 - \rho)^{\sum_{k \in \mathbb{Z}_n} (1 - \eta(k))}$$

is invariant for the WASEP (this is a good exercise!). In the following we will consider the stationary WASEP $(\eta^n_t)_{t \geqslant 0}$ on $\mathbb{Z}_n$ with initial distribution $\eta^n_0 \sim \mu^n_{1/2}$ (which we denote by μ^n in the following) and we will show, following the approach developed by [GoncalvesJara], that under the right rescaling it converges to a FB-solution of Burgers equation. The restriction to $\rho = 1/2$ is for convenience, the same works for all $\rho \in (0,1)$, but for $\rho = 1/2$ some things simplify.

Next, let us identify the symmetric part of the generator and the antisymmetric part, to understand how the process behaves under time reversal:

Lemma 6 *The symmetric and antisymmetric part of $\mathcal{L}^n$ in $L^2(\mu^n)$ are, respectively*

$$\mathcal{L}^n_S f(\eta) = \frac{1}{2} \sum_{k \in \mathbb{Z}_n} (f(\eta^{k,k+1}) - f(\eta)),$$

$$\mathcal{L}^n_A f(\eta) = \frac{n^{-1/2}}{2} \sum_{k \in \mathbb{Z}_n} [\eta(k) - \eta(k+1)](f(\eta^{k,k+1}) - f(\eta)).$$

Moreover, the time reversed process is again a WASEP, only that the asymmetry is in the opposite direction.

Proof Let $f, g : S_n \to \mathbb{R}$. Since $\eta^{k,\ell}$ has the same distribution as η under μ^n, we get

$$\begin{aligned} E[f\mathcal{L}^n g] &= \sum_{k,\ell\in\mathbb{Z}_n} \int f(\eta)\eta(k)(1-\eta(\ell))p_n(\ell-k)(g(\eta^{k,\ell})-g(\eta))\mu^n(\mathrm{d}\eta) \\ &= \sum_{k,\ell\in\mathbb{Z}_n} \Bigg[\int f(\eta^{\ell,k})\eta^{\ell,k}(k)(1-\eta^{\ell,k}(\ell))p_n(\ell-k)g(\eta)\mu^n(\mathrm{d}\eta) \\ &\qquad - \int f(\eta)\eta(k)(1-\eta(\ell))p_n(\ell-k)g(\eta)\mu^n(\mathrm{d}\eta)\Bigg]. \end{aligned}$$

Now we use that $\eta^{\ell,k}(k) = \eta(\ell)$ and $\eta^{\ell,k}(\ell) = \eta(k)$ and exchange the names of the summation variables k and ℓ to obtain

$$\begin{aligned} E[f\mathcal{L}^n g] &= \sum_{k,\ell\in\mathbb{Z}_n} \Bigg[\int f(\eta^{\ell,k})\eta(\ell)(1-\eta(k))p_n(\ell-k)g(\eta)\mu^n(\mathrm{d}\eta) \\ &\qquad - \int f(\eta)\eta(k)(1-\eta(\ell))p_n(\ell-k)g(\eta)\mu^n(\mathrm{d}\eta)\Bigg] \\ &= \sum_{k,\ell\in\mathbb{Z}_n} \int \eta(k)(1-\eta(\ell))p_n(k-\ell)(f(\eta^{k,\ell})-f(\eta))g(\eta)\mu^n(\mathrm{d}\eta) \\ &= \sum_{k,\ell\in\mathbb{Z}_n} \int \eta(k)(1-\eta(\ell))p_n^*(\ell-k)(f(\eta^{k,\ell})-f(\eta))g(\eta)\mu^n(\mathrm{d}\eta), \end{aligned}$$

where we used that

$$\begin{aligned} \sum_{k,\ell\in\mathbb{Z}_n} \eta(k)(1-\eta(\ell))(p_n(\ell-k)-p_n(k-\ell)) &= \sum_{k,\ell\in\mathbb{Z}_n} \eta(k)(p_n(\ell-k)-p_n(k-\ell)) \\ &= \sum_{k\in\mathbb{Z}_n} \eta(k)(1-1) = 0, \end{aligned}$$

and where

$$p_n^*(k) = p_n(-k) = \begin{cases} \frac{1}{2}(1+n^{-1/2}), & k=-1, \\ \frac{1}{2}(1-n^{-1/2}), & k=1, \\ 0, & \text{else}. \end{cases}$$

So $\mathcal{L}^{n,*}$ is defined as $\mathcal{L}^n$, only with p_n^* instead of p_n, and therefore

$$\mathcal{L}_S^n f(\eta) = \frac{1}{2}(\mathcal{L}^n + \mathcal{L}^{n,*})f(\eta) = \sum_{k,\ell\in\mathbb{Z}_n} \eta(k)(1-\eta(\ell))p_{n,S}(\ell-k)(f(\eta^{k,\ell})-f(\eta)),$$

where

$$p_{n,S}(k) = \frac{1}{2}(p_n(k)+p_n^*(k)) = \begin{cases} \frac{1}{2}, & k\in\{-1,1\}, \\ 0, & \text{else}. \end{cases}$$

Since $p_{n,S}(k) = p_{n,S}(-k)$ and $\eta^{k,\ell} = \eta^{\ell,k}$, we get

$$\begin{aligned}
\mathcal{L}_S^n f(\eta) &= \frac{1}{2} \sum_{k,\ell \in \mathbb{Z}_n} [\eta(k)(1-\eta(\ell)) + \eta(\ell)(1-\eta(k))] p_{n,S}(\ell - k)(f(\eta^{k,\ell}) - f(\eta)) \\
&= \frac{1}{2} \sum_{k,\ell \in \mathbb{Z}_n} p_{n,S}(\ell - k)(f(\eta^{k,\ell}) - f(\eta)) \\
&= \frac{1}{2} \sum_{k \in \mathbb{Z}_n} \left[\frac{1}{2}(f(\eta^{k,k+1}) - f(\eta)) + \frac{1}{2}(f(\eta^{k,k-1}) - f(\eta)) \right] \\
&= \frac{1}{2} \sum_{k \in \mathbb{Z}_n} (f(\eta^{k,k+1}) - f(\eta)),
\end{aligned}$$

where we used that $\eta(k)(1-\eta(\ell)) + \eta(\ell)(1-\eta(k)) = 1$ whenever $\eta(k) \neq \eta(\ell)$, and that otherwise $f(\eta^{k,\ell}) - f(\eta) = 0$. Similarly, the antisymmetric part of the generator is

$$\begin{aligned}
\mathcal{L}_A^n f(\eta) &= \frac{1}{2}(\mathcal{L}^n - \mathcal{L}^{n,*}) f(\eta) = \sum_{k,\ell \in \mathbb{Z}_n} \eta(k)(1-\eta(\ell)) p_{n,A}(\ell - k)(f(\eta^{k,\ell}) - f(\eta)), \\
&= \frac{n^{-1/2}}{2} \sum_{k \in \mathbb{Z}_n} \eta(k)(1-\eta(k+1))(f(\eta^{k,k+1}) - f(\eta)) \\
&\quad - \frac{n^{-1/2}}{2} \sum_{k \in \mathbb{Z}_n} \eta(k)(1-\eta(k-1))(f(\eta^{k,k-1}) - f(\eta)) \\
&= \frac{n^{-1/2}}{2} \sum_{k \in \mathbb{Z}_n} [\eta(k)(1-\eta(k+1)) - \eta(k+1)(1-\eta(k))](f(\eta^{k,k+1}) - f(\eta)) \\
&= \frac{n^{-1/2}}{2} \sum_{k \in \mathbb{Z}_n} [\eta(k) - \eta(k+1)](f(\eta^{k,k+1}) - f(\eta)),
\end{aligned}$$

where

$$p_{n,A}(k) = \frac{1}{2}(p_n(k) - p_n^*(k)) = \begin{cases} -\frac{n^{-1/2}}{2}, & k = -1, \\ \frac{n^{-1/2}}{2}, & k = 1, \\ 0, & \text{else}, \end{cases}$$

(and of course this is no longer a transition probability). □

To understand the behavior of the WASEP on large scales, we rescale space to bring $\mathbb{Z}_n$ into the torus $\mathbb{T}$. We are interested in the invariant measure, and using the (proof of) the weak law of large numbers it is easy to see that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k \in \mathbb{Z}_n} \eta_t^n(k) \varphi\left(\frac{k}{n}\right) = \frac{1}{2} \int_{\mathbb{T}} \varphi(x) \mathrm{d}x$$

for all $t > 0$ and $\varphi \in C^\infty(\mathbb{T})$ (even for all $\varphi \in C(\mathbb{T})$), where the factor $1/2$ is the occupation density of μ^n. We want to show that on an appropriate time scale (the one under which each randomly walking particle would converge to a Brownian motion if it was not for the exclusion rule) the fluctuations around the density $1/2$ converge to Burgers equation. So we set

$$\mathcal{Y}_t^n(\varphi) = \frac{1}{n} \sum_{k \in \mathbb{Z}_n} n^{1/2} \left(\eta_{n^2 t}^n(k) - \frac{1}{2} \right) \varphi\left(\frac{k}{n}\right) = n^{-1/2} \sum_{k \in \mathbb{Z}_n} \left(\eta_{n^2 t}^n(k) - \frac{1}{2} \right) \varphi\left(\frac{k}{n}\right).$$

Since for fixed $t \geqslant 0$ the variables $(\eta^n_{n^2t}(k)-1/2)_{k\in\mathbb{Z}_n}$ are i.i.d., centered, and have variance $1/4$, we get from the (Lindeberg) central limit theorem that $\mathcal{Y}^n_t(\varphi)$ converges to a $\mathcal{N}(0,\|\varphi\|^2_{L^2}/4)$ variable. In other words, for all times $t \geqslant 0$ the limit of $\mathcal{Y}^n_t$ is $1/2$ times a white noise. So we see that we are definitely in a different parameter range than in the previous chapters. But this is no problem, we could have easily adapted the arguments to prove the uniqueness of FB-solutions with general parameters.

Obviously $\mathcal{Y}^n \in D(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}))$, where D is the Skorohod space, equipped with the Skorohod topology (see e.g. [EK86]). We want to prove tightness of $\mathcal{Y}^n$ and that every limiting point is supported in $C(\mathbb{R}_+, \mathcal{S}'(\mathbb{T}))$ and is a FB-solution of Burgers equation. For proving tightness it suffices to show that $(\mathcal{Y}^n_t(\varphi))_{t\geqslant 0}$ is tight for all $\varphi \in C^\infty(\mathbb{T})$ [Mit83], and we get from the martingale problem for η^n:

$$\begin{aligned}\mathcal{Y}^n_t(\varphi) - \mathcal{Y}^n_0(\varphi) &= \int_0^{n^2t} \mathcal{L}^n F^{n,\varphi}(\eta^n_s)\mathrm{d}s + M^{n,\varphi}_{n^2t} \\ &= \int_0^t n^2 \mathcal{L}^n F^{n,\varphi}(\eta^n_{n^2s})\mathrm{d}s + \mathcal{M}^n_t(\varphi),\end{aligned}$$

where

$$F^{n,\varphi}(\eta) = 2n^{-1/2} \sum_{k\in\mathbb{Z}_n} \left(\eta(k) - \frac{1}{2}\right)\varphi\left(\frac{k}{n}\right)$$

and

$$\mathcal{M}^n_t = M^{n,\varphi}_{n^2t}$$

is a martingale in the filtration generated by $(\eta^n_{n^2t})_{t\geqslant 0}$, which is the same as the filtration generated by $\mathcal{Y}^n$. Note that

$$\begin{aligned}F^{n,\varphi}(\eta^{\ell,\ell+1}) - F^{n,\varphi}(\eta) &= n^{-1/2}\sum_{k\in\mathbb{Z}_n}\left(\eta^{\ell,\ell+1}(k) - \frac{1}{2}\right)\varphi\left(\frac{k}{n}\right) - n^{-1/2}\sum_{k\in\mathbb{Z}_n}\left(\eta(k)-\frac{1}{2}\right)\varphi\left(\frac{k}{n}\right) \\ &= n^{-1/2}\Bigg[\sum_{k\in\mathbb{Z}_n}\left(\eta(k)-\frac{1}{2}\right)\varphi\left(\frac{k}{n}\right) \\ &\qquad + \left(\left(\eta(\ell+1)-\frac{1}{2}\right) - \left(\eta(\ell)-\frac{1}{2}\right)\right)\varphi\left(\frac{\ell}{n}\right) \\ &\qquad + \left(\left(\eta(\ell)-\frac{1}{2}\right) - \left(\eta(\ell+1)-\frac{1}{2}\right)\right)\varphi\left(\frac{\ell+1}{n}\right) \\ &\qquad - \sum_{k\in\mathbb{Z}_n}\left(\eta(k)-\frac{1}{2}\right)\varphi\left(\frac{k}{n}\right)\Bigg] \\ &= -n^{-1/2}\left(\left(\eta(\ell+1)-\frac{1}{2}\right) - \left(\eta(\ell)-\frac{1}{2}\right)\right)\left(\varphi\left(\frac{\ell+1}{n}\right) - \varphi\left(\frac{\ell}{n}\right)\right).\end{aligned}$$

We decompose the drift into its symmetric and antisymmetric part:

$$
\begin{aligned}
\mathcal{L}_S^n F^{n,\varphi}(\eta) &= \frac{1}{2}\sum_{\ell\in\mathbb{Z}_n}(F^{n,\varphi}(\eta^{\ell,\ell+1}) - F^{n,\varphi}(\eta)) \\
&= -\frac{n^{-1/2}}{2}\sum_{\ell\in\mathbb{Z}_n}\left(\left(\eta(\ell+1)-\frac{1}{2}\right)-\left(\eta(\ell)-\frac{1}{2}\right)\right)\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right) \\
&= \frac{n^{-1/2}}{2}\sum_{\ell\in\mathbb{Z}_n}\left(\eta(\ell)-\frac{1}{2}\right)\left[\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right)-\left(\varphi\left(\frac{\ell}{n}\right)-\varphi\left(\frac{\ell-1}{n}\right)\right)\right] \\
&= \frac{1}{2}n^{-1/2}\sum_{\ell\in\mathbb{Z}_n}\left(\eta(\ell)-\frac{1}{2}\right)n^{-2}\Delta_n\varphi\left(\frac{\ell}{n}\right),
\end{aligned}
$$

where

$$
\Delta_n\varphi(x) = n^2\left(\varphi\left(x+\frac{1}{n}\right)+\varphi\left(x-\frac{1}{n}\right)-2\varphi(x)\right)
$$

is the discrete Laplacian, rescaled so that it converges to the continuous Laplacian as $n\to\infty$. Therefore,

$$
\int_0^{n^2 t}\mathcal{L}_S^n F^{n,\varphi}(\eta_s^n)\mathrm{d}s = \frac{1}{2}\int_0^t \mathcal{Y}_s^n(\Delta_n\varphi)\mathrm{d}s,
$$

which already looks good. For the antisymmetric part we get

$$
\begin{aligned}
\mathcal{L}_A^n F^{n,\varphi}(\eta) &= \frac{n^{-1/2}}{2}\sum_{\ell\in\mathbb{Z}_n}[\eta(\ell)-\eta(\ell+1)](F^{n,\varphi}(\eta^{\ell,\ell+1}) - F^{n,\varphi}(\eta)) \\
&= \frac{n^{-1}}{2}\sum_{\ell\in\mathbb{Z}_n}[\eta(\ell)-\eta(\ell+1)]\left(\left(\eta(\ell+1)-\frac{1}{2}\right)-\left(\eta(\ell)-\frac{1}{2}\right)\right)\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right) \\
&= -\frac{n^{-1}}{2}\sum_{\ell\in\mathbb{Z}_n}(\bar\eta(\ell+1)-\bar\eta(\ell))^2\left[\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right] \\
&= -\frac{n^{-1}}{2}\sum_{\ell\in\mathbb{Z}_n}\left(\frac{1}{4}-2\bar\eta(\ell)\bar\eta(\ell+1)+\frac{1}{4}\right)\left[\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right] \\
&= n^{-1}\sum_{\ell\in\mathbb{Z}_n}\bar\eta(\ell)\bar\eta(\ell+1)\left[\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right] \\
&= n^{-2}\sum_{\ell\in\mathbb{Z}_n}\bar\eta(\ell)\bar\eta(\ell+1)\nabla_n\varphi\left(\frac{\ell}{n}\right) \\
&= n^{-2}\sum_{\ell\in\mathbb{Z}_n}\bar\eta(\ell)\bar\eta(\ell+1)\nabla_n\varphi\left(\frac{\ell}{n}\right)
\end{aligned}
$$

where the contribution from the two 1/4 terms vanishes because it is a telescope sum with periodic boundary conditions, where $\bar\eta(k)=\eta(k)-\frac{1}{2}$, and where we used that $\bar\eta(k)^2 = 1/4$ for all k (it is here that we need $\rho = 1/2$, all other steps work for general $\rho\in(0,1)$), and

$$
\nabla_n\varphi(x) = n\left(\varphi\left(x+\frac{1}{n}\right)-\varphi(x)\right)
$$

is the discrete derivative, rescaled so that it converges to the continuous one. Therefore,

$$\mathcal{Y}_t^n(\varphi) - \mathcal{Y}_0^n(\varphi) = \frac{1}{2}\int_0^t \mathcal{Y}_s^n(\Delta_n\varphi)\mathrm{d}s + A_t^n(\varphi) + \mathcal{M}_t^n(\varphi),$$

where

$$\begin{aligned} A_t^n(\varphi) &= \int_0^t \sum_{\ell\in\mathbb{Z}_n} \bar{\eta}^n_{n^2s}(\ell)\bar{\eta}^n_{n^2s}(\ell+1)\nabla_n\varphi\left(\frac{\ell}{n}\right)\mathrm{d}s \\ &= \int_0^t n^{-1}\sum_{\ell\in\mathbb{Z}_n}[n^{1/2}\bar{\eta}^n_{n^2s}(\ell)][n^{1/2}\bar{\eta}^n_{n^2s}(\ell+1)]\nabla_n\varphi\left(\frac{\ell}{n}\right)\mathrm{d}s. \end{aligned}$$

By Dynkin's formula, the martingale $\mathcal{M}^n(\varphi)$ has predictable quadratic variation

$$\langle\mathcal{M}^n(\varphi)\rangle_t = \int_0^t [n^2\mathcal{L}^n(F^{n,\varphi})^2(\eta^n_{n^2s}) - 2F^{n,\varphi}(\eta^n_{n^2s})n^2\mathcal{L}^nF^{n,\varphi}(\eta^n_{n^2s})]\mathrm{d}s,$$

where we already computed $\mathcal{L}^nF^{n,\varphi}(\eta)$ and where by the same arguments

$$\begin{aligned} (F^{n,\varphi})^2(\eta^{\ell,\ell+1}) - (F^{n,\varphi})^2(\eta) &= \left(n^{-1/2}\sum_{k\in\mathbb{Z}_n}\bar{\eta}^{\ell,\ell+1}(k)\varphi\left(\frac{k}{n}\right)\right)^2 \\ &\quad - \left(n^{-1/2}\sum_{k\in\mathbb{Z}_n}\bar{\eta}(k)\varphi\left(\frac{k}{n}\right)\right)^2 \\ &= n^{-1}\left[\sum_{k\in\mathbb{Z}_n}\bar{\eta}(k)\varphi\left(\frac{k}{n}\right) - (\bar{\eta}(\ell+1)-\bar{\eta}(\ell))\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right)\right]^2 \\ &\quad - n^{-1}\left[\sum_{k\in\mathbb{Z}_n}\bar{\eta}(k)\varphi\left(\frac{k}{n}\right)\right]^2 \\ &= -2n^{-1}(\bar{\eta}(\ell+1)-\bar{\eta}(\ell))\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right)\sum_{k\in\mathbb{Z}_n}\bar{\eta}(k)\varphi\left(\frac{k}{n}\right) \\ &\quad + n^{-1}(\bar{\eta}(\ell+1)-\bar{\eta}(\ell))^2\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right)^2 \end{aligned}$$

and thus

$$\begin{aligned} \mathcal{L}_S^n(F^{n,\varphi})^2(\eta) &= -\frac{1}{2}\sum_{\ell\in\mathbb{Z}_n}2n^{-1}(\bar{\eta}(\ell+1)-\bar{\eta}(\ell))\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right)\sum_{k\in\mathbb{Z}_n}\bar{\eta}(k)\varphi\left(\frac{k}{n}\right) \\ &\quad + \frac{1}{2}\sum_{\ell\in\mathbb{Z}_n}n^{-1}(\bar{\eta}(\ell+1)-\bar{\eta}(\ell))^2\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right)^2 \\ &= n^{-5/2}\sum_{\ell\in\mathbb{Z}_n}\bar{\eta}(\ell)\Delta_n\varphi\left(\frac{\ell}{n}\right)F^{n,\varphi}(\eta) + \frac{n^{-3}}{2}\sum_{\ell\in\mathbb{Z}_n}(\bar{\eta}(\ell+1)-\bar{\eta}(\ell))^2\left(\nabla_n\varphi\left(\frac{\ell}{n}\right)\right)^2 \\ &= 2F^{n,\varphi}(\eta)\mathcal{L}_S^nF^{n,\varphi}(\eta) + \frac{n^{-3}}{2}\sum_{\ell\in\mathbb{Z}_n}(\bar{\eta}(\ell+1)-\bar{\eta}(\ell))^2\left(\nabla_n\varphi\left(\frac{\ell}{n}\right)\right)^2, \end{aligned}$$

while

$$\begin{aligned}
\mathcal{L}_A^n(F^{n,\varphi})^2 &= -\frac{n^{-1/2}}{2}\sum_{\ell\in\mathbb{Z}_n}[\eta(\ell)-\eta(\ell+1)]2n^{-1}(\bar\eta(\ell+1)-\bar\eta(\ell)) \\
&\qquad \times\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right)\sum_{k\in\mathbb{Z}_n}\bar\eta(k)\varphi\left(\frac{k}{n}\right) \\
&\quad+\frac{n^{-1/2}}{2}\sum_{\ell\in\mathbb{Z}_n}[\eta(\ell)-\eta(\ell+1)]n^{-1}(\bar\eta(\ell+1)-\bar\eta(\ell))^2\left(\varphi\left(\frac{\ell+1}{n}\right)-\varphi\left(\frac{\ell}{n}\right)\right)^2 \\
&= n^{-2}\sum_{\ell\in\mathbb{Z}_n}(\bar\eta(\ell+1)-\bar\eta(\ell))^2\nabla_n\varphi\left(\frac{\ell}{n}\right)n^{-1/2}\sum_{k\in\mathbb{Z}_n}\bar\eta(k)\varphi\left(\frac{k}{n}\right) \\
&\quad-\frac{n^{-7/2}}{2}\sum_{\ell\in\mathbb{Z}_n}(\bar\eta(\ell+1)-\bar\eta(\ell))^3\left(\nabla_n\varphi\left(\frac{\ell}{n}\right)\right)^2 \\
&= 2F^{n,\varphi}(\eta)\mathcal{L}_A^nF^{n,\varphi}(\eta)-\frac{n^{-7/2}}{2}\sum_{\ell\in\mathbb{Z}_n}(\bar\eta(\ell+1)-\bar\eta(\ell))^3\left(\nabla_n\varphi\left(\frac{\ell}{n}\right)\right)^2.
\end{aligned}$$

So overall

$$\begin{aligned}
\mathrm{d}\langle\mathcal{M}^n(\varphi)\rangle_t &= [n^2\mathcal{L}^n(F^{n,\varphi})^2(\eta^n_{n^2t})-2F^{n,\varphi}(\eta^n_{n^2t})n^2\mathcal{L}^nF^{n,\varphi}(\eta^n_{n^2t})]\mathrm{d}t \\
&= \frac{n^{-1}}{2}\sum_{\ell\in\mathbb{Z}_n}(\bar\eta(\ell+1)-\bar\eta(\ell))^2[1-n^{-1/2}(\bar\eta(\ell+1)-\bar\eta(\ell))]\left(\nabla_n\varphi\left(\frac{\ell}{n}\right)\right)^2\mathrm{d}t.
\end{aligned}$$

Note that

$$\int(\bar\eta(\ell+1)-\bar\eta(\ell))^2\mu^n(\mathrm{d}\eta)=\frac{1}{4}-0+\frac{1}{4}=\frac{1}{2},$$

and that $(\bar\eta(\ell+1)-\bar\eta(\ell))^2$ and $(\bar\eta(k+1)-\bar\eta(k))^2$ are independent whenever $|k-\ell|>1$. Based on all these observations, the following lemma is fairly easy to show:

Lemma 7 *The processes*

$$\frac{1}{2}\int_0^{\cdot}\mathcal{Y}_s^n(\Delta_n\varphi)\mathrm{d}s, \qquad \mathcal{M}^n(\varphi)$$

are tight in $D(\mathbb{R}_+,\mathbb{R})$ and their limit points are all supported in $C(\mathbb{R}_+,\mathbb{R})$. If $\mathcal{Y}^n$ converges in distribution in $D(\mathbb{R}_+,\mathcal{S}'(\mathbb{T}))$ along a subsequence to some $\mathcal{Y}$, then $\frac{1}{2}\int_0^{\cdot}\mathcal{Y}_s^n(\Delta_n\varphi)\mathrm{d}s$ converges along the same subsequence to $\frac{1}{2}\int_0^{\cdot}\mathcal{Y}(\Delta\varphi)\mathrm{d}s$. Moreover, $\mathcal{M}^n(\varphi)$ converges in distribution to a continuous martingale $\mathcal{M}(\varphi)$ with quadratic variation

$$\langle\mathcal{M}(\varphi)\rangle_t=\frac{1}{4}\int_{\mathbb{T}}|\partial_x\varphi(x)|^2\mathrm{d}x.$$

To conclude the proof, we only have to study what happens with A^n. This is much more complicated, and [GJ14] developed new techniques to deal with that term. Note that

$$\begin{aligned}
A_t^n(\varphi) &= \int_0^t n^{-1}\sum_{\ell\in\mathbb{Z}_n}[n^{1/2}\bar\eta^n_{n^2s}(\ell)][n^{1/2}\bar\eta^n_{n^2s}(\ell+1)]\nabla_n\varphi\left(\frac{\ell}{n}\right)\mathrm{d}s \\
&= \int_0^t n^{-1}\sum_{\ell\in\mathbb{Z}_n}\mathcal{Y}_s^n\left(\frac{\ell}{n}\right)\mathcal{Y}_s^n\left(\frac{\ell+1}{n}\right)\nabla_n\varphi\left(\frac{\ell}{n}\right)\mathrm{d}s
\end{aligned}$$

"looks" very much like the square of $\mathcal{Y}^n$ integrated against $\partial_x\varphi$, but of course evaluating $\mathcal{Y}^n$ pointwise is not a continuous operation in $\mathcal{S}'(\mathbb{T})$, and neither is taking the product. We therefore want to approximate the ill-defined functional by a well defined functional: If we could show tightness of $A^n(\varphi)$ and that uniformly in n

$$\mathbb{E}\left[\left|A^n_t(\varphi)-A^n_s(\varphi)-\int_s^t n^{-1}\sum_{\ell\in\mathbb{Z}_n}\mathcal{Y}^n_s\left(\delta_\varepsilon\left(\frac{\ell}{n}-\cdot\right)\right)^2\nabla_n\varphi\left(\frac{\ell}{n}\right)\mathrm{d}s\right|^2\right]\leqslant C\varepsilon(t-s),\qquad(9)$$

then we would get from Fatou's lemma that any limit point $A(\varphi)$ has to satisfy

$$\mathbb{E}\left[\left|A_t(\varphi)-A_s(\varphi)-\int_s^t\int_{\mathbb{T}}\mathcal{Y}_s(\delta_\varepsilon(x-\cdot))^2\partial_x\varphi(x)\mathrm{d}s\right|^2\right]\leqslant C\varepsilon(t-s).$$

If δ_ε is an approximation of the identity, then we could conclude that any limit point $\mathcal{Y}$ is a martingale solution to

$$\partial_t\mathcal{Y}=\frac{1}{2}\Delta u-\partial_x u^2+\frac{1}{2}\partial_x\xi.$$

But actually the quantitative estimate for A is also sufficient to see that the quadratic variation of A must be zero, so we even get an energy solution. Moreover, we already computed the symmetric and antisymmetric part of the generator and saw that the time-reversed process is a WASEP with opposite sign of the asymmetry, and therefore $\mathcal{Y}$ is even an FB-solution.

To prove (9) we use again the Itô trick, as in the previous chapter. Then we can either solve the corresponding Poisson equation, which is possible because we have a chaos expansion for our product Bernoulli invariant measure, or we can use the Kipnis-Varadhan approach and estimate the $\mathcal{H}^{-1}$ norm. This is the approach taken by [GJ14], who replace $\mathcal{Y}^n_s(\ell)\mathcal{Y}^n_s(\ell+1)$ by the product of averages over successively larger boxes (the boxes double in size in each step) and write the difference $A^n_t(\varphi)-A^n_s(\varphi)-\int_s^t n^{-1}\sum_{\ell\in\mathbb{Z}_n}\mathcal{Y}^n_s\left(\delta_\varepsilon\left(\frac{\ell}{n}-\cdot\right)\right)^2\nabla_n\varphi\left(\frac{\ell}{n}\right)\mathrm{d}s$ as a telescope sum and estimate each term using $\mathcal{H}^{-1}$ techniques.

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