A SUPERHEDGING APPROACH TO STOCHASTIC INTEGRATION

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Abstract. Using Vovk's outer measure, which corresponds to a minimal superhedging price, the existence of quadratic variation is shown for "typical price paths" in the space of non-negative càdlàg functions. In particular, this implies the existence of quadratic variation in the sense of Föllmer quasi surely under all martingale measures.

Based on the robust existence of quadratic variation and a certain topology which is induced by Vovk's outer measure, model-free Itô integration is developed on the space of continuous paths, of non-negative càdlàg paths and of càdlàg paths with mildly restricted jumps.

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1. Introduction

In a recent series of papers [Vov09, Vov12, Vov15], Vovk introduced a model-free, hedging-based approach to mathematical finance that uses arbitrage considerations to examine which path properties are satisfied by "typical price paths". For this purpose an outer measure $\overline{P}$ is introduced, which corresponds to a superhedging price, and a statement (A) is said to hold for "typical price paths" if (A) is true except a null set under $\overline{P}$. Let us also refer to [SV01] and [TKT09] for related settings. As a nice consequence all results proven for typical price paths hold quasi surely under all martingale measures and even more generally quasi surely under all semimartingale measures for which the coordinate process satisfies the classical condition of "no arbitrage of the first kind", also known as "no unbounded profit with bounded risk".

Considering typical price paths belonging to the space $C([0,T],\mathbb{R}^d)$ of continuous functions, their path properties are already rather well-studied. Vovk proved that the path regularity of typical price paths is similar to that of semimartingales (see [Vov08]), i.e. non-constant typical price paths have infinite p-variation for $p<2$ but finite p-variation for $p>2$, and they possess a quadratic variation (see [Vov12]). More advanced path properties such as the existence of an associated local time or an Itô rough path were shown in [PP15] and [PP16]. All these results give a robust justification for taking auxiliary properties of continuous semimartingales as an underlying assumption in model-independent finance or mathematical finance under Knightian uncertainty.

However, while in financial modeling stochastic processes allowing for jumps play a central role, typical price paths belonging to the space $D([0,T],\mathbb{R}^d)$ of càdlàg functions are not

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well-understood yet. This turns out to be a more delicate business because of particular two reasons: First there exists no canonical extension of Vovk's outer measure $\overline{P}$ to the whole space $D([0,T],\mathbb{R}^d)$ as discussed in Remark 2.2 and thus we need to work with suitable subspaces. Second, the class of admissible strategies gets smaller if the class of possible price trajectories gets bigger.

The main motivation of the presented article is to develop a rather complete picture of "stochastic" Itô integration of adapted càglàd integrands with respect to typical price paths. For this purpose we consider the underlying spaces of continuous functions $C([0,T],\mathbb{R}^d)$, of non-negative càdlàg functions $D([0,T],\mathbb{R}^d_+)$, and of càdlàg functions with mildly restrict jumps Ω_ψ, that is the subset of all $\omega \in D([0,T],\mathbb{R}^d)$ such that

$$|\omega(t)-\omega(t-)| \leq \psi\Big(\sup_{s\in[0,t)}|\omega(s)|\Big), \quad t\in(0,T],$$

where $\omega(t-) := \lim_{s\to t,\, s<t}\omega(s)$ and $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a fixed non-decreasing function. The financial interpretation of this last space is discussed in detail in [Vov15].

Our first contribution is to prove the existence of quadratic variation for typical non-negative càdlàg price paths, which is constructed as the uniform limit of discrete versions of the quadratic variation. For the precise result we refer to Theorem 3.2 and Corollary 3.11. Intuitively, this means that it should be possible to make an arbitrarily large profit by investing in those paths where the convergence of the discrete quadratic variation fails. Let us emphasize once again that this, in particular, justifies to assume the existence of the quadratic variation in model-free financial mathematics. In the case of continuous paths or càdlàg paths with restricted jumps, the existence of the quadratic variation for typical price paths has been shown by Vovk in [Vov12] and [Vov15], respectively. However, the case of non-negative càdlàg paths stayed open as it is rather tricky in the sense that here no short-selling is allowed and therefore a priori the arguments used in [Vov12, Vov15] break down.

The existence of the quadratic variation is not only a crucial ingredient to develop a model-free Itô integration, it is also a powerful assumption in robust and model-independent financial mathematics since it allows to use Föllmer's pathwise Itô integral [Föl79] and thus to construct $\int f'(\omega(s))\,\mathrm{d}\omega(s)$ for $f\in C^2$ or more generally for path-dependent functionals in the sense of Dupire [Dup09] as shown by Cont and Fournié [CF10]. Föllmer integration has a long history of applications in mathematical finance going as least back to Bick and Willinger [BW94] and Lyons [Lyo95] and is still a frequently used tool as one can see in many recent works, e.g. [DOR14, SV15, Rig16] and the references therein.

Our second contribution is the development of a model-free Itô integration theory for adapted càglàd integrands with respect to typical price paths, see Theorem 4.2. Compared to the rich list of pathwise constructions of stochastic integrals, e.g. [Bic81], [WT88], [WT89], [Kar95], [Nut12], the presented construction complements the previous works in three aspects:

(1) It works without any tools from probability theory at all and is entirely based on pathwise superhedging arguments.
(2) The non-existence of the Itô integral comes with a natural arbitrage interpretation, that is, it is possible to achieve an arbitrage opportunity of the first kind if the Itô integral does not exist.
(3) The model-free Itô integral possesses good continuity estimates (in a topology induced by Vovk's outer measure $\overline{P}$).

In the case of continuous paths the main ingredient for our construction is a continuity estimate for the pathwise Itô integral of a step function, see Lemma 4.3. This estimate is based on Vovk's pathwise Hoeffding inequality ([Vov12, Theorem A.1]) and the existence of the quadratic variation. In this context, we recover essentially the results of [PP16, Theorem 3.5]. However, in [PP16] we worked with the uniform topology on the space of integrands while here we are able to strengthen our results and to replace the uniform distance with a rather natural distance that depends only on the integral of the squared integrand against the quadratic variation.

In the case of càdlàg paths the constructions of the model-free Itô integral also rely on continuity estimates for the integral of a step functions, see Lemmas 4.5 and 4.7, but this time the estimates require completely different techniques compared to the continuous case. In particular, Vovk's pathwise Hoeffding inequality needs to be replaced by a pathwise version of the Burkholder-Davis-Gundy inequality due to Beiglböck and Siorpaes [BS15], which again takes for granted the existence of the quadratic variation.

Organization of the paper. Section 2 introduces Vovk's model-free and hedging-based approach to mathematical finance. In Section 3 the existence of quadratic variation for typical non-negative càdlàg price paths is shown. The model-free Itô integration is developed in Section 4. Appendix A collects some auxiliary results concerning Vovk's outer measure.

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2. Superhedging and typical price paths

Vovk's model-free and hedging-based approach to mathematical finance allows for determining sample path properties of "typical price paths". For this purpose he introduces a notion of outer measure which is based on purely pathwise arbitrage considerations, see for example [Vov12]. Following a slightly modified framework as introduced in [PP16, PP15], we briefly set up the notation and definitions.

For $d \in \mathbb{N}$ and a real number $T \in (0, \infty)$ we consider the following three different (sample) spaces of possible price paths:

- $\Omega_c := C([0,T], \mathbb{R}^d)$ stands for the space of all continuous functions $\omega\colon [0,T] \to \mathbb{R}^d$.
- $\Omega_+ := D_{S_0}([0,T], \mathbb{R}^d_+)$ is the space of all non-negative càdlàg functions $\omega\colon [0,T] \to \mathbb{R}^d_+$ such that $\omega(0) = S_0$ for some fixed $S_0 \in \mathbb{R}^d_+$ and $\mathbb{R}_+ := [0, \infty)$.
- Ω_ψ denotes the subset of all càdlàg functions $\omega\colon [0,T] \to \mathbb{R}^d$ such that

$$|\omega(t) - \omega(t-)| \leq \psi\Big(\sup_{s\in[0,t)} |\omega(s)|\Big), \quad t \in (0,T],$$

  where $\omega(t-) := \lim_{s\to t,\, s<t} \omega(s)$ and $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a fixed non-decreasing function.

In the subsequent we sometimes just write Ω meaning that it is possible to chose Ω as any of the above sample spaces Ω_c, Ω_+ or Ω_ψ. The space of all càdlàg functions $\omega\colon [0,T] \to \mathbb{R}^d$ is denoted by $D([0,T], \mathbb{R}^d)$.

For each $t \in [0,T]$, $\mathcal{F}^\circ_t$ is defined to be the smallest σ-algebra Ω that makes all functions $\omega \mapsto \omega(s)$, $s \in [0,t]$, measurable and $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}^\circ_t$. An event is an element of the σ-algebra $\mathcal{F}_T$. Stopping times $\tau\colon \Omega \to [0,T] \cup \{\infty\}$ with respect to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual. The coordinate process on Ω is denoted by S, i.e. $S_t(\omega) := \omega(t)$ for $\omega \in \Omega$ and $t \in [0,T]$. The indicator function of a set A, for $A \subset \mathbb{R}$ or $A \subset \Omega$, is denoted by $\mathbf{1}_A$ and for two real vectors $x, y \in \mathbb{R}^d$ we write $xy = x \cdot y$ for the inner product on $(\mathbb{R}^d, |\cdot|)$, where $|\cdot|$ denotes the Euclidean norm. Furthermore, we set $s \wedge t := \min\{s,t\}$ and $s \vee t := \max\{s,t\}$ for $s,t \in \mathbb{R}_+$.

A process $H \colon \Omega \times [0,T] \to \mathbb{R}^d$ is a *simple (trading) strategy* if there exist a sequence of stopping times $0 = \tau_0 < \tau_1 < \dots$, and $\mathcal{F}_{\tau_n}$-measurable bounded functions $h_n \colon \Omega \to \mathbb{R}^d$, such that for every $\omega \in \Omega$, $\tau_n(\omega) = \tau_{n+1}(\omega) = \ldots \in [0,\infty]$ from some n on, and such that

$$H_t(\omega) = \sum_{n=0}^{\infty} h_n(\omega) \mathbf{1}_{(\tau_n(\omega), \tau_{n+1}(\omega)]}(t), \quad t \in [0,T].$$

Therefore, for a simple strategy H the corresponding integral process

$$(H \cdot S)_t(\omega) := \sum_{n=0}^{\infty} h_n(\omega) \cdot (S_{\tau_{n+1}\wedge t}(\omega) - S_{\tau_n \wedge t}(\omega)) = \sum_{n=0}^{\infty} h_n(\omega) S_{\tau_n \wedge t, \tau_{n+1}\wedge t}(\omega)$$

is well-defined for all $\omega \in \Omega$ and all $t \in [0,T]$; here we introduced the notation $S_{u,v} := S_v - S_u$ for $u, v \in [0,T]$.

For $\lambda > 0$ a simple strategy H is called *(strongly) λ-admissible* if $(H \cdot S)_t(\omega) \geq -\lambda$ for all $\omega \in \Omega$ and all $t \in [0,T]$. The set of strongly λ-admissible simple strategies is denoted by $\mathcal{H}_\lambda$.

In the next definition we introduce an outer measure $\overline{P}$, which is very similar to the one used by Vovk [Vov12], but not quite the same. We refer to [PP16, Section 2.3] for a detailed discussion of the relation between our slightly modified outer measure and the original one due to Vovk.

Definition 2.1. *Vovk's outer measure* $\overline{P}$ of a set $A \subseteq \Omega$ is defined as the minimal superhedging price for $\mathbf{1}_A$, that is

$$\overline{P}(A) := \inf \Big\{ \lambda > 0 \, : \, \exists (H^n)_{n \in \mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \forall \omega \in \Omega \, : \, \liminf_{n \to \infty} (\lambda + (H^n \cdot S)_T(\omega)) \geq \mathbf{1}_A(\omega) \, \Big\}.$$

A set $A \subseteq \Omega$ is called a *null set* if it has outer measure zero. A property (P) holds for *typical price paths* if the set A where (P) is violated is a null set.

Remark 2.2. *Even if it might be desirable to consider the more general sample space $\Omega = D([0,T], \mathbb{R}^d)$ of all càdlàg functions as possible price paths, there seems to be no natural extension of Vovk's outer measure to this setting. Therefore, it is necessary to distinguish between the space Ω_ψ of càdlàg functions with mildly restricted jumps and the space Ω_+ of non-negative càdlàg functions starting from a fixed initial value.*

Indeed, as in [Vov12, Lemma 4.1] or [PP16, Lemma 2.3], it is straightforward to verify that $\overline{P}$ fulfills all properties of an outer measure.

Lemma 2.3. *$\overline{P}$ is an outer measure, i.e. a non-negative set function defined on the subsets of Ω such that $\overline{P}(\emptyset) = 0$, $\overline{P}(A) \leq \overline{P}(B)$ for $A \subseteq B \subset \Omega$, and $\overline{P}(\cup_n A_n) \leq \sum_n \overline{P}(A_n)$ for every sequence of subsets $(A_n)_{n \in \mathbb{N}} \subset \Omega$.*

One of the main reasons why Vovk's outer measure $\overline{P}$ is of such great interest in model-independent financial mathematics, is that it dominates simultaneously all local martingale measures on the sample space Ω (cf. [Vov12, Lemma 6.3] and [PP16, Proposition 2.6]). In other words, a null set under $\overline{P}$ turns out to be a null set simultaneously under all local martingale measures on Ω.

Proposition 2.4. *Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$, such that the coordinate process S is a $\mathbb{P}$-local martingale, and let $A \in \mathcal{F}_T$. Then $\mathbb{P}(A) \leq \overline{P}(A)$.*

A second reason is that sets with outer measure zero come with a natural arbitrage interpretation from classical mathematical finance. Roughly speaking, a null set leads to a pathwise arbitrage opportunity of the first kind (NA1), see [Vov12, p. 564] and [PP16, Lemma 2.4].

Proposition 2.5. *A set $A \subseteq \Omega$ is a null set if and only if there exists a constant $K \in (0, \infty)$ and a sequence of K-admissible simple strategies $(H^n)_{n\in\mathbb{N}} \subset \mathcal{H}_K$ such that*

$$\liminf_{n\to\infty} (K + (H^n \cdot S)_T(\omega)) \geq \infty \cdot \mathbf{1}_A(\omega) \tag{1}$$

with the conventions $0 \cdot \infty = 0$ and $1 \cdot \infty = \infty$.

As the proofs of Proposition 2.4 and Proposition 2.5 work exactly as in the case of continuous price paths, they are postponed to the Appendix A.

2.1. **Auxiliary set function for non-negative paths.** In order to prove the existence of the quadratic variation for typical non-negative price paths, we need to introduce a relaxed version of admissibility. Throughout this subsection we only consider $\Omega_+ = D_{S_0}([0,T], \mathbb{R}^d_+)$.

A trading strategy H is called a *weakly λ-admissible* strategy if H is a simple strategy such that

$$(H \cdot S)_t(\omega) \geq -\lambda(1 + \mathbb{1} \cdot S_{\rho_\lambda(H)}(\omega)\mathbf{1}_{[\rho_\lambda(H)(\omega),T]}(t))$$

with $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^d$ and $H_t(\omega) = H_t(\omega)\mathbf{1}_{[0,\rho_\lambda(H)(\omega)\wedge T]}(t)$, where

$$\rho_\lambda(H)(\omega) := \inf\{t \in [0,T] \,:\, (H \cdot S)_t(\omega) < -\lambda\},$$

for all $(t, \omega) \in [0,T] \times \Omega_+$, with the conventions $\inf \emptyset := \infty$, $[\infty, T] := \emptyset$ and $S_\infty(\omega) := 0$. We write $\mathcal{G}_\lambda$ for the set of weakly λ-admissible strategies. Expressed verbally, this means that weakly λ-admissible strategies must give a payoff larger than $-\lambda$ at all times, except that they can lose all their previous gains plus $\lambda(1 + \mathbb{1} \cdot S_t)$ through one large jump; however, in that case they must instantly stop trading and may not try to bounce up again.

Based on the notion of weak admissibility, we define an auxiliary set function $\overline{Q}$ via the minimal superhedging price using this lager class of trading strategies.

Definition 2.6. The set function $\overline{Q}$ is given by

$$\overline{Q}(A) := \inf\Big\{\lambda > 0 \,:\exists (H^n)_{n\in\mathbb{N}} \subseteq \mathcal{G}_\lambda \text{ s.t. } \forall \omega \in \Omega_+ : \\ \liminf_{n\to\infty} (\lambda + (H^n \cdot S)_T(\omega) + \lambda \mathbb{1} \cdot S_{\rho_\lambda(H^n)}(\omega)\mathbf{1}_{\{\rho_\lambda(H^n)<\infty\}}(\omega)) \geq \mathbf{1}_A(\omega)\Big\},$$

where $\{\rho_\lambda(H^n) < \infty\} := \{\omega \in \Omega_+ \,:\, \rho_\lambda(H^n)(\omega) < \infty\}$.

Remark 2.7. *Note that $\overline{Q}$ is not an outer measure since it fails to be even finitely subadditive. However, it will be a useful tool to show the existence of the quadratic variation for typical non-negative càdlàg price paths, and the following lemma (Lemma 2.8) shows that it is* nearly *subadditive:*

$$\overline{Q}(\cup_n A_n) \leq \overline{P}(\cup_n A_n) \leq \sum_n \overline{P}(A_n) \leq (1 + \mathbb{1} \cdot S_0) \sum_n \overline{Q}(A_n),$$

for every sequence $(A_n)_{n\in\mathbb{N}} \subset \Omega_+$. Hence, in the particular case where $\overline{Q}(A_n) = 0$ for all $n \in \mathbb{N}$ we still get the countable subadditivity.

Lemma 2.8. *We have* $\overline{Q}(A) \le \overline{P}(A) \le (1 + \mathbb{1} \cdot S_0)\overline{Q}(A)$ *for all* $A \subset \Omega_+$.

Proof. The first inequality is clear. For the second one, assume that $\overline{Q}(A) < \lambda$ for $A \subset \Omega_+$. Then there exists a sequence $(G^n)_{n\in\mathbb{N}} \subset \mathcal{G}_\lambda$ such that

$$\liminf_{n\to\infty} (\lambda + (G^n \cdot S)_T + \lambda \mathbb{1} \cdot S_{\rho_\lambda(G^n)} \mathbf{1}_{\{\rho_\lambda(G^n)<\infty\}}) \ge \mathbf{1}_A. \tag{2}$$

Then the strategy $H^n := \mathbf{1}_{(0,\rho_\lambda(G^n)]}\lambda\mathbb{1} + G^n$ satisfies for $t < \rho_\lambda(G^n)$

$$\lambda(1 + \mathbb{1} \cdot S_0) + (H^n \cdot S)_t = \lambda(1 + \mathbb{1} \cdot S_0) + (G^n \cdot S)_t + \lambda\mathbb{1} \cdot (S_t - S_0) \ge \lambda\mathbb{1} \cdot S_t \ge 0,$$

and for $t = \rho_\lambda(G^n)$

$$\begin{aligned}\lambda(1 + \mathbb{1} \cdot S_0) + (H^n \cdot S)_{\rho_\lambda(G^n)} &= \lambda(1 + \mathbb{1} \cdot S_0) + (G^n \cdot S)_{\rho_\lambda(G^n)} + \lambda\mathbb{1} \cdot (S_{\rho_\lambda(G^n)} - S_0) \\ &\ge \lambda - \lambda(1 + \mathbb{1} \cdot S_{\rho_\lambda(G^n)}) + \lambda\mathbb{1} \cdot S_{\rho_\lambda(G^n)} = 0,\end{aligned}$$

which then extends to $(\rho_\lambda(G^n), T]$ because both G^n and $\mathbf{1}_{[0,\rho_\lambda(G^n)]}$ vanish on that interval. This implies $H^n \in \mathcal{H}_{\lambda(1+\mathbb{1}\cdot S_0)}$, and moreover from (2) we get

$$\begin{aligned}\liminf_{n\to\infty} (\lambda(1 + \mathbb{1} \cdot S_0)&+(H^n \cdot S)_T) \\ &= \liminf_{n\to\infty} (\lambda(1 + \mathbb{1} \cdot S_0) + (G^n \cdot S)_T + \lambda\mathbb{1} \cdot (S_{\rho_\lambda(G^n)\wedge T} - S_0)) \ge \mathbf{1}_A.\end{aligned}$$

Hence, $\overline{P}(A) \le (1 + \mathbb{1} \cdot S_0)\lambda$, which proves our claim. □

3. Quadratic variation for non-negative càdlàg price paths

The existence of quadratic variation for typical price paths is the crucial ingredient in the construction of model-free Itô integration. While the existence was already proven by Vovk in the case of continuous price paths and price paths with mildly restricted jumps (see [Vov12] and [Vov15]), the situation for non-negative càdlàg price paths was completely unclear so far.

In this section we show that the quadratic variation along suitable sequences of partitions exists for typical non-negative price paths without any restriction on the allowed jumps. We first restrict our considerations to one-dimensional paths and consider $\Omega_+ = D_{S_0}([0,T], \mathbb{R}_+)$. The extension to $\Omega_+ = D_{S_0}([0,T], \mathbb{R}^d_+)$ for arbitrary $d \in \mathbb{N}$ is given in Subsection 3.1.

Definition 3.1. Let $n \in \mathbb{N}$ and let $\mathbb{D}^n := \{k2^{-n} : k \in \mathbb{Z}\}$ be the dyadic numbers. For a real-valued càdlàg function $\omega\colon [0,T] \to \mathbb{R}$ its *Lebesgue partition* $\pi_n(\omega) := \{\tau^n_k(\omega) : k \ge 0\}$ is given by the sequence of stopping times $(\tau^n_k(\omega))_{k\in\mathbb{N}}$ inductively defined by

$$\tau^n_0(\omega) := 0 \quad \text{and} \quad D^n_0(\omega) := \sup(\mathbb{D}^n \cap (-\infty, S_0(\omega)]),$$

and for every $k \in \mathbb{N}$ we further set

$$\begin{aligned}\tau^n_k(\omega) &:= \inf\{t \in [\tau^n_{k-1}(\omega), T] : [\![S_{\tau^n(\omega)_{k-1}}(\omega), S_t(\omega)]\!] \cap (\mathbb{D}^n \setminus \{D^n_{k-1}(\omega)\}) \ne \emptyset\}, \\ D^n_k(\omega) &:= \operatorname{argmin}_{D\in[\![S_{\tau^n_{k-1}(\omega)}(\omega), S_{\tau^n_k(\omega)}(\omega)]\!]\cap(\mathbb{D}^n\setminus\{D^n_{k-1}(\omega)\})} |D - S_{\tau^n_k(\omega)}(\omega)|,\end{aligned}$$

with the convention $\inf \emptyset = \infty$ and

$$[\![u, v]\!] := \begin{cases} [u, v] & \text{if } u \le v, \\ [v, u] & \text{if } u > v. \end{cases}$$

Notice that $D_k^n(\omega)$, $k \in \mathbb{N}$, are $\mathcal{F}_{\tau_k^n}$-measurable functions and they are uniquely determined since $D_k^n(\omega)$ is a one-element set for each $k \in \mathbb{N}$. In the following we often just write τ_k^n and π_n instead of $\tau_k^n(\omega)$ and π_n, respectively.

Along the sequence of Lebesgue partitions we obtain the existence of quadratic variation for typical non-negative price paths.

Theorem 3.2. *For typical price paths $\omega \in \Omega_+ = D_{S_0}([0,T],\mathbb{R}_+)$ the discrete quadratic variation*

$$Q_t^n(\omega) := \sum_{k=1}^{\infty} (S_{\tau_k^n \wedge t}(\omega) - S_{\tau_{k-1}^n \wedge t}(\omega))^2, \quad t \in [0,T], \tag{3}$$

along the Lebesgue partitions $(\pi_n(\omega))_{n\in\mathbb{N}}$ converges in the uniform metric to a function $[S](\omega) \in D([0,T],\mathbb{R}_+)$.

Remark 3.3. *Since by* [Vov15, Lemma 3] *the sequence of partitions $(\pi_n)_{n\in\mathbb{N}}$ is increasing and exhausts $S(\omega)$,* [Vov15, Lemma 2] *states that the limit $[S](\omega)$ will be a non-decreasing càdlàg function satisfying $[S]_0(\omega) = 0$ and $\Delta[S]_t(\omega) = (\Delta S_t(\omega))^2$ for all $t \in (0,T]$ and all $\omega \in \Omega_+$ for which the convergence in Theorem 3.2 holds. Here we used the notation $\Delta f_t := f_t - \lim_{s\to t,\, s<t} f_s$ for $f \in D([0,T],\mathbb{R})$.*

In order to prove Theorem 3.2, we first analyze the crossings behavior of typical price paths with respect to the dyadic levels $\mathbb{D}^n$. To be more precise, we introduce the number of upcrossings (resp. downcrossings) of a function f over an open interval $(a,b) \subset \mathbb{R}$.

Definition 3.4. Let $f\colon [0,T] \to \mathbb{R}$ be a càdlàg function, $(a,b) \subset \mathbb{R}$ be an open non-empty interval and $t \in [0,T]$. The number $\mathrm{U}_t^{(a,b)}(f)$ of upcrossings of the interval (a,b) by the function f during the time interval $[0,t]$ is given by

$$\mathrm{U}_t^{(a,b)}(f) := \sup_{n\in\mathbb{N}} \sup_{0\le s_1<t_1<\cdots<s_n<t_n\le t} \sum_{i=1}^{n} I(f(s_i), f(t_i)),$$

where

$$I(f(s_i), f(t_i)) := \begin{cases} 1 & \text{if } f(s_i) \le a \text{ and } f(t_i) \ge b, \\ 0 & \text{if otherwise.} \end{cases}$$

The number $\mathrm{D}_t^{(a,b)}(f)$ of downcrossings is analogously defined as the number of upcrossings.

For each $h > 0$ we also introduce the accumulated number of upcrossing resp. downcrossings by

$$\mathrm{U}_t(f,h) := \sum_{k\in\mathbb{Z}} \mathrm{U}_t^{(kh,(k+1)h)}(f) \quad \text{and} \quad \mathrm{D}_t(f,h) := \sum_{k\in\mathbb{Z}} \mathrm{D}_t^{(kh,(k+1)h)}(f).$$

In order to provide a deterministic inequality in the spirit of Doob's upcrossing lemma, we use the stopping times

$$\gamma_K(\omega) := \inf\{t \in [0,T] \,:\, S_t(\omega) \ge K\}$$

for $\omega \in \Omega_+$ and $K \in \mathbb{N}$.

Lemma 3.5. *Let $K \in \mathbb{N}$. For each $n \in \mathbb{N}$, there exists a strongly 1-admissible simple strategy $H^n \in \mathcal{H}_1$ such that*

$$1 + (H^n \cdot S)_{t\wedge\gamma_K}(\omega) \ge 2^{-2n} K^{-2} \mathrm{U}_{t\wedge\gamma_K}(\omega, 2^{-n})$$

for all $t \in [0,T]$ and every $\omega \in \Omega_+ = D_{S_0}([0,T],\mathbb{R}_+)$.

Proof. Let us start by considering the upcrossings $\mathrm{U}_t^{(a,b)}(\omega)$ of an interval $(a,b) \subseteq [0,K]$. By buying one unit the first time $S_t(\omega)$ drops below a and selling the next time $S_t(\omega)$ goes above b and continuing in this manner, we obtain a simple strategy $H^{(a,b)} \in \mathcal{H}_a$ with

$$a + (H^{(a,b)} \cdot S)_t(\omega) \geq (b-a)\mathrm{U}_t^{(a,b)}(\omega), \quad t \in [0,T], \omega \in \Omega_+. \tag{4}$$

For a formal construction of $H^{(a,b)}$ we refer to [Vov15, Lemma 4.5]. Note that we need the non-negativity of the paths in Ω_+ to guarantee the strong admissibility of $H^{(a,b)}$. Set now

$$H^n := 2^{-n}K^{-2} \sum_{k\in\mathbb{N}_0,\,(k+1)2^{-n}\leq K} H^{(k2^{-n},(k+1)2^{-n})}.$$

Since $H^{(k2^{-n},(k+1)2^{-n})} \in \mathcal{H}_{k2^{-n}} \subset \mathcal{H}_K$ for all k with $(k+1)2^{-n} \leq K$, we have $H^n \in \mathcal{H}_1$, and

$$1 + (H^n \cdot S)_t(\omega) \geq 2^{-2n}K^{-2} \sum_{k\in\mathbb{N}_0,\,(k+1)2^{-n}\leq K} \mathrm{U}_t^{(k2^{-n},(k+1)2^{-n})}(\omega), \quad (t,\omega) \in [0,T]\times\Omega_+,$$

which proves the claim because $\mathrm{U}_{t\wedge\gamma_K}(\omega, 2^{-n}) = \sum_{k\in\mathbb{N}_0,\,(k+1)2^{-n}\leq K} \mathrm{U}_{t\wedge\gamma_K}^{(k2^{-n},(k+1)2^{-n})}(\omega)$. □

With this pathwise version of Doob's upcrossing lemma at hand, we can control the number of level crossings of typical non-negative price paths.

Corollary 3.6. *For typical price paths $\omega \in \Omega_+ = D_{S_0}([0,T],\mathbb{R}_+)$ there exist an $N(\omega) \in \mathbb{N}$ such that*

$$\mathrm{U}_T(\omega, 2^{-n}) \leq n^2 2^{2n} \quad \text{and} \quad \mathrm{D}_T(\omega, 2^{-n}) \leq n^2 2^{2n}$$

for all $n \geq N(\omega)$.

Proof. Since for each $k \in \mathbb{Z}$, $\mathrm{U}_t^{(k2^{-n},(k+1)2^{-n})}(\omega)$ and $\mathrm{D}_t^{(k2^{-n},(k+1)2^{-n})}(\omega)$ differ by no more than 1, we have $|\mathrm{U}_T(\omega,2^{-n}) - \mathrm{D}_T(\omega,2^{-n})| \in [0, 2^n K]$ for all $n \in \mathbb{N}$ and for every $\omega \in \Omega_+$ with $\sup_{t\in[0,T]}|S_t(\omega)| < K$. So if we show that $\overline{P}(B_K) = 0$ for all $K \in \mathbb{N}$, where $B_K := \cap_{m\in\mathbb{N}} \cup_{n\geq m} A_{K,n}$ with

$$A_{K,n} = \left\{ \omega \in \Omega_+ : \sup_{t\in[0,T]} |S_t(\omega)| < K \text{ and } \mathrm{U}_T(\omega, 2^{-n}) \geq \frac{n^2 2^{2n}}{2} \right\},$$

then our claim follows from the countable subadditivity of $\overline{P}$. But using Lemma 3.5 we immediately obtain that $\overline{P}(A_{K,n}) \leq n^{-2} 2K^2$, and since this is summable, it suffices to apply the Borel-Cantelli lemma (see Lemma A.1) to see $\overline{P}(B_K) = 0$. □

To prove the convergence of the discrete quadratic variation processes $(Q^n)_{n\in\mathbb{N}}$, we shall show that the sequence $(Q^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the uniform topology on Ω_+. For this purpose, we define the auxiliary sequence $(Z^n)_{n\in\mathbb{N}}$ by

$$Z_t^n := Q_t^n - Q_t^{n-1}, \quad t \in [0,T].$$

Similarly as in Vovk [Vov15], the proof of Theorem 3.2 is based on the sequence of integral processes $(\mathcal{K}^n)_{n\in\mathbb{N}}$ given by

$$\mathcal{K}_t^n := n^4 2^{-2n} + n^4 2^{-n+4} K^2 + (Z_t^n)^2 - \sum_{k=1}^{\infty} (Z^n_{\tau_k^n \wedge t} - Z^n_{\tau_{k-1}^n \wedge t})^2, \quad t \in [0,T], \tag{5}$$

for $K \in \mathbb{N}$, and the stopping times

$$\sigma_K^n := \min\left\{\tau_k^n : \sum_{i=1}^k (Z^n_{\tau_i^n} - Z^n_{\tau_{i-1}^n})^2 > n^4 2^{-2n}\right\} \wedge \min\{\tau_k^n : Z^n_{\tau_k^n} > K\}, \quad n \in \mathbb{N}. \tag{6}$$

The next lemma presents that each $\mathcal{K}^n$ is indeed an integral process with respect to a weakly admissible simple strategy, cf. [Vov15, Lemma 5].

Lemma 3.7. *For each $n \in \mathbb{N}$ and $K \in \mathbb{N}$, there exists a weakly admissible simple strategy $L^{K,n} \in \mathcal{G}_{n^4 2^{-2n} + n^4 2^{-n+4} K^2}$ such that*

$$\mathcal{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t} = n^4 2^{-2n} + n^4 2^{-n+4} K^2 + (L^{K,n} \cdot S)_t, \quad t \in [0,T].$$

Proof. For each $K \in \mathbb{N}$ and each $n \in \mathbb{N}$ [Vov15, Lemma 5] shows the equality for the strategy

$$L_t^{K,n} := \mathbf{1}_{(0,\gamma_K \wedge \sigma_K^n]}(t) \sum_k (-4) Z^n_{\tau_k^n} (S_{\tau_k^n} - S_{\chi^{n-1}(\tau_k^n)}) \mathbf{1}_{(\tau_k^n, \tau_{k+1}^n]}(t), \quad t \in [0,T],$$

where $\chi^{n-1}(t) := \max\{\tau_{k'}^{n-1} : \tau_{k'}^{n-1} \le t\}$. Since $L^{K,n}$ is obviously a simple strategy, it remains to prove that $L^{K,n} \in \mathcal{G}_{n^4 2^{-2n} + n^4 2^{-n+4} K^2}$.

First we observe up to time $\tilde{\tau}^n := \max\{\tau_k^n : \tau_k^n < \gamma_K \wedge \sigma_K^n\}$ that

$$\min_{t \in [0, \tilde{\tau}^n]} \mathcal{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t} \ge n^4 2^{-n+4} K^2, \tag{7}$$

which follows directly from the definition of $\mathcal{K}^n$ and (6). For $t \in (\tilde{\tau}^n, \gamma_K \wedge \sigma_K^n]$ notice that

$$|S_{\tilde{\tau}^n} - S_{\chi^{n-1}(\tilde{\tau}^n)}| \le 2^{-n+2}, \tag{8}$$

since we either have $\tilde{\tau}^n \in \pi_{n-1}$, which implies $\chi^{n-1}(\tilde{\tau}^n) = \tilde{\tau}^n$ and $|S_{\tilde{\tau}^n} - S_{\chi^{n-1}(\tilde{\tau}^n)}| = 0$, or we have $\tilde{\tau}^n \notin \pi_{n-1}$, which implies (8) as $\tilde{\tau}^n < \tau^{n-1}_{k'+1}$ and

$$|S_{\tilde{\tau}^n} - S_{\chi^{n-1}(\tilde{\tau}^n)}| \le |S_{\tilde{\tau}^n} - D^{n-1}_{k'}| + |S_{\chi^{n-1}(\tilde{\tau}^n)} - D^{n-1}_{k'}| \le 2^{-n+1} + 2^{-n+1},$$

where k' is such that $\chi^{n-1}(\tilde{\tau}^n) = \tau^{n-1}_{k'}$. Using (8), $|Z^n_{\tilde{\tau}^n}| \le K$ and $|S_{\tilde{\tau}^n}| \le K$, we estimate

$$\begin{aligned} |4 Z^n_{\tilde{\tau}^n} (S_{\tilde{\tau}^n} - S_{\chi^{n-1}(\tilde{\tau}^n)})(S_t - S_{\tilde{\tau}^n})| &\le 4K 2^{-n+2} |S_t - S_{\tilde{\tau}^n}| \\ &\le 4K 2^{-n+2}(S_t + K) \le n^4 2^{-n+4} K S_t + n^4 2^{-n+4} K^2, \end{aligned} \tag{9}$$

which together with (7) and $K \le K^2$ gives us the claimed weak admissibility, $L^{K,n} \in \mathcal{G}_{n^4 2^{-2n} + n^4 2^{-n+4} K^2}$. □

Corollary 3.8. *For typical price paths $\omega \in \Omega_+ = D_{S_0}([0,T], \mathbb{R}_+)$ there exist an $N(\omega) \in \mathbb{N}$ such that*

$$\mathcal{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t}(\omega) < n^6 2^{-n}, \quad t \in [0,T],$$

for all $n \ge N(\omega)$.

Proof. Consider the events

$$A_n := \left\{\omega \in \Omega_+ : \exists t \in [0,T] \text{ s.t. } \mathcal{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t}(\omega) \ge n^6 2^{-n}\right\}, \qquad n \in \mathbb{N}.$$

By the Borel-Cantelli lemma (see Lemma A.1) the claim follows once we have shown that $\sum_n \overline{P}(A_n) < \infty$. To that end, we define the stopping times

$$\rho^n := \inf\{t \in [0,T] : \mathcal{K}^n_{\gamma_K \wedge \sigma_K^n \wedge t} \ge n^6 2^{-n}\}, \quad n \in \mathbb{N},$$

so that

$$A_n = \Big\{\omega \in \Omega_+ \,:\, n^{-6} 2^n \mathcal{K}^n_{\gamma_K \wedge \sigma^n_K \wedge \rho^n \wedge T}(\omega) \geq 1\Big\}.$$

Now it follows directly from Lemma 3.7 that

$$\overline{Q}(A_n) \leq n^{-6} 2^n (n^4 2^{-2n} + n^4 2^{-n+4} K^2) = n^{-2} 2^{-n} + n^{-2} 2^4 K^2,$$

which is summable. Since $\overline{P}(A_n) \leq (1+S_0)\overline{Q}(A_n)$ by Lemma 2.8, the proof is complete. □

Finally, we have collected all necessary ingredients to prove the main result of this section, namely Theorem 3.2. More precisely, we shall show that $(Q^n - Q^{n-1})_{n\in\mathbb{N}}$ is a Cauchy sequence. This implies Theorem 3.2 since the uniform topology on Ω_+ is complete.

Proof of Theorem 3.2. For $K \in \mathbb{N}$ let us define

$$A_K := \Big\{\omega \in \Omega_+ \,:\, \sup_{t\in[0,T]} |S_t(\omega)| \leq K \text{ and } \sup_{t\in[0,T]} |Z^n_t(\omega)| \geq n^3 2^{-\frac{n}{2}} \text{ for infinitely many } n \in \mathbb{N}\Big\}$$

and

$$B := \Big\{\omega \in \Omega_+ \,: \exists N(\omega) \in \mathbb{N} \text{ s.t. } \mathcal{K}^n_{\gamma_K \wedge \sigma^n_K \wedge t}(\omega) < n^6 2^{-n}, \quad t \in [0,T],$$
$$\mathrm{U}_T(\omega, 2^{-n}) \leq n^2 2^{2n} \text{ and } \mathrm{D}_T(\omega, 2^{-n}) \leq n^2 2^{2n}, \quad n \geq N(\omega)\Big\}.$$

Thanks to the countable subadditivity of $\overline{P}$ it is sufficient to show that $\overline{P}(A_K) = 0$ for every $K \in \mathbb{N}$. Moreover, again by the subadditivity of $\overline{P}$ we see

$$\overline{P}(A_K) \leq \overline{P}(A_K \cap B) + \overline{P}(A_K \cap B^c).$$

By Corollary 3.6 and Corollary 3.8 it is already known that $\overline{P}(A_K \cap B^c) = 0$. In the following we show that $A_K \cap B = \emptyset$.

For this purpose, let us fix an $\omega \in B$ such that $\sup_{t\in[0,T]} |S_t(\omega)| \leq K$. Since $\omega \in B$ there exits an $N(\omega) \in \mathbb{N}$ such that for all $m \geq N(\omega)$:

(a) The number of stopping times in π_m does not exceed $2m^2 2^{2m} + 2 \leq 3m^2 2^{2m}$.
(b) The number of stopping times in π_m such that

$$|\Delta S_{\tau^m_k}(\omega)| := \Big| S_{\tau^m_k}(\omega) - \lim_{s\to\tau^m_k,\, s<\tau^m_k} S_s(\omega)\Big| \geq 2^{-m+1}, \quad \tau^m_k \in \pi_m,$$

is less or equal to $2m^2 2^{2m}$.

As $\sup_{t\in[0,T]} |S_t(\omega)| \leq K$, notice that $\gamma_K(\omega) = T$ and that for $t \in [0,T]$ we have

$$\begin{aligned} Z^n_{\tau^n_{k+1}\wedge t}(\omega) - Z^n_{\tau^n_k \wedge t}(\omega) &= (Q^n_{\tau^n_{k+1}\wedge t}(\omega) - Q^n_{\tau^n_k\wedge t}(\omega)) - (Q^{n-1}_{\tau^n_{k+1}\wedge t}(\omega) - Q^{n-1}_{\tau^n_k\wedge t}(\omega)) \\ &= (S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k \wedge t}(\omega))^2 \\ &\quad - ((S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\chi^{n-1}(\tau^n_k\wedge t)}(\omega))^2 - (S_{\tau^n_k\wedge t}(\omega) - S_{\chi^{n-1}(\tau^n_k \wedge t)}(\omega))^2) \\ &= -2(S_{\tau^n_k\wedge t}(\omega) - S_{\chi^{n-1}(\tau^n_k\wedge t)}(\omega))(S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k\wedge t}(\omega)), \end{aligned}$$

where we recall that $\chi^{n-1}(t) := \max\{\tau_{k'}^{n-1} \,:\, \tau_{k'}^{n-1} \leq t\}$. Therefore, keeping (8) in mind, the infinite sum in (5) can be estimated by

$$\sum_{k=0}^{\infty}\left(Z^n_{\tau^n_{k+1}\wedge t}(\omega) - Z^n_{\tau^n_k\wedge t}(\omega)\right)^2 = 4\sum_{k=0}^{\infty}(S_{\tau^n_k\wedge t}(\omega) - S_{\chi^{n-1}(\tau^n_k\wedge t)}(\omega))^2(S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k\wedge t}(\omega))^2$$

$$\leq 2^{6-2n}\sum_{k=0}^{\infty}(S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k\wedge t}(\omega))^2. \tag{10}$$

For $n \geq N = N(\omega)$ and $t \in [0,T]$ we observe for the summands in (10) the following bounds, which are similar to the bounds (A)-(E) in the proof of [Vov15, Theorem 1]:

(1) If $\tau^n_{k+1} \notin \pi_{n-1}$, then one has $\chi^{n-1}(\tau^n_{k+1}) = \chi^{n-1}(\tau^n_k) = \tau^{n-1}_{k'}$ for some k' and thus

$$|S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k\wedge t}(\omega)| \leq |S_{\tau^n_{k+1}\wedge t}(\omega) - D^{n-1}_{k'}| + |S_{\tau^n_k\wedge t}(\omega) - D^{n-1}_{k'}| \leq 2^{2-n}.$$

The number of such summands is at most $3n^2 2^{2n}$.

(2) If $\tau^n_{k+1} \in \pi_{n-1}$ and $|\Delta S_{\tau^n_{k+1}}| \leq 2^{-n+1}$, then one has

$$|S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k\wedge t}(\omega)| \leq 2^{1-n} + 2^{-n+1} = 2^{2-n}$$

and the number of such summands is at most $3n^2 2^{2n}$.

(3) If $\tau^n_{k+1} \in \pi_{n-1}$ and $|\Delta S_{\tau^n_{k+1}}| \in [2^{-m+1}, 2^{-m+2})$, for some $m \in \{N, N+1, \dots, n\}$ than one has that

$$|S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k\wedge t}(\omega)| \leq 2^{1-n} + 2^{-m+2}.$$

and the number of such summands is at most $2m^2 2^{2m}$.

(4) If $\tau^n_{k+1} \in \pi_{n-1}$ and $\Delta S_{\tau^n_{k+1}} \geq 2^{-N+2}$, then one has

$$|S_{\tau^n_{k+1}\wedge t}(\omega) - S_{\tau^n_k(\omega)\wedge t}| \leq K$$

and the number of such summand is bounded by a constant $C = C(\omega, K)$ independent of n.

Using the bounds derived in (1)-(4), the estimate (10) can be continued by

$$\sum_{k=0}^{\infty}\left(Z^n_{\tau^n_{k+1}\wedge t}(\omega) - Z^n_{\tau^n_k\wedge t}(\omega)\right)^2 \leq 2^{6-2n}\left(6n^2 2^{2n} 2^{4-2n} + \sum_{m=N}^{n} 2m^2 2^{2m}(2^{1-n} + 2^{-m+2})^2 + CK^2\right),$$

and thus there exists an $\tilde{N} = \tilde{N}(\omega) \in \mathbb{N}$ such that

$$\sum_{k=0}^{\infty}\left(Z^n_{\tau^n_{k+1}\wedge t}(\omega) - Z^n_{\tau^n_k\wedge t}(\omega)\right)^2 \leq 2^{-2n}n^4, \quad t \in [0,T],$$

for all $n \geq \tilde{N}$. Combining the last estimate with the definition of $\mathcal{K}^n$ (cf. (5)), we obtain

$$\mathcal{K}^n_{\sigma^n_K\wedge t}(\omega) \geq (Z^n_{\sigma^n_K\wedge t}(\omega))^2, \quad t \in [0,T],$$

for all $n \geq \tilde{N}$. Moreover, by assumption on ω one has

$$\mathcal{K}^n_{\sigma^n_K\wedge t}(\omega) < n^6 2^{-n}, \quad t \in [0,T],$$

for all $n \geq N \vee \tilde{N}$. In particular, we conclude that $\sup_{t\in[0,T]} |Z^n_{\sigma^n_K \wedge \gamma_K \wedge t}(\omega)| < K$ whenever n is large enough and thus

$$n^6 2^{-n} > (Z^n_t(\omega))^2, \quad t \in [0,T],$$

for all sufficiently large n. Finally, we have $\sup_{t\in[0,T]}|Z^n_t(\omega)| < n^3 2^{-\frac{n}{2}}$ for all large n and therefore $\omega \notin A_K \cap B$. □

Remark 3.9. *The existence of quadratic variation in the sense of Theorem 3.2 is equivalent to the existence of quadratic variation in the sense of Föllmer (see* [Vov15, Proposition 3]*). Therefore, Theorem 3.2 opens the door to apply Föllmer's pathwise Itô formula* [Föl79] *for typical non-negative price paths and in particular to define the pathwise integral $\int f'(S_s)\,\mathrm{d}S_s$ for $f \in C^2$ or more general for path-dependent functionals as shown by Cont and Fournié* [CF10]*, Imkeller and Prömel* [IP15]*, and Ananova and Cont* [AC16]*.*

3.1. **Extension to multi-dimensional price paths.** In order to extend the existence of quadratic variation from one-dimensional to multi-dimensional typical price paths, we consider the sample space $\Omega_+ = D([0,T],\mathbb{R}^d_+)$ and introduce a d-dimensional version of the Lebesgue partitions for $d \in \mathbb{N}$.

Definition 3.10. For $n \in \mathbb{N}$ and a d-dimensional càdlàg function $\omega\colon [0,T] \to \mathbb{R}^d$ its *Lebesgue partition* $\pi_n(\omega) := \{\tau^n_k(\omega) : k \geq 0\}$ is iteratively defined by $\tau^n_0(\omega) := 0$ and

$$\tau^n_k(\omega) := \min\left\{\tau > \tau^n_{k-1}(\omega) \,:\, \tau \in \bigcup_{i=1}^d \pi_n(\omega^i) \cup \bigcup_{i,j=1, i\neq j}^d \pi_n(\omega^i + \omega^j)\right\}, \quad k \in \mathbb{N},$$

where $\omega = (\omega^1,\dots,\omega^d)$ and $\pi_n(\omega^i)$ and $\pi_n(\omega^i+\omega^j)$ are the Lebesgue partitions of ω^i and $\omega^i+\omega^j$ as introduced in Definition 3.1, respectively.

To state the existence of quadratic variation for typical price paths in Ω_+, we define the canonical projection on Ω_+ by $S^i_t(\omega) = \omega^i(t)$ for $\omega = (\omega^1,\dots,\omega^d) \in \Omega_+$, $t \in [0,T]$ and $i = 1,\dots,d$.

Corollary 3.11. *Let $d \in \mathbb{N}$ and $1 \leq i,j \leq d$. For typical price paths $\omega \in \Omega_+ = D([0,T],\mathbb{R}^d_+)$ the discrete quadratic variation*

$$Q^{i,j,n}_t(\omega) := \sum_{k=1}^{\infty} (S^i_{\tau^n_k \wedge t}(\omega) - S^i_{\tau^n_{k-1}\wedge t}(\omega))(S^j_{\tau^n_k \wedge t}(\omega) - S^j_{\tau^n_{k-1}\wedge t}(\omega)), \quad t \in [0,T],$$

converges along the Lebesgue partitions $(\pi_n(\omega))_{n\in\mathbb{N}}$ in the uniform metric to a function $[S^i,S^j](\omega) \in D([0,T],\mathbb{R}_+)$.

Proof. To show the convergence of $Q^{i,j,n}_\cdot(\omega)$ for a path $\omega \in \Omega_+$, we observe that

$$\begin{aligned}\frac{1}{2}&(((S^i_t(\omega)+S^j_t(\omega)) - (S^i_s(\omega)+S^j_s(\omega)))^2 - (S^i_t(\omega)-S^i_s(\omega))^2 - (S^j_t(\omega)-S^j_s(\omega))^2)\\ &= (S^i_t(\omega)-S^i_s(\omega))(S^j_t(\omega)-S^j_s(\omega)), \quad s,t \in [0,T],\end{aligned}$$

and thus it is sufficient to prove the existence of the quadratic variation of $S^i(\omega)$ and $S^i(\omega)+S^j(\omega)$ for $1 \leq i,j \leq d$ with $i \neq j$. For typical price paths this can be done precisely as in the proof of Theorem 3.2 with the only exception that the bounds (a)-(b) and (1)-(4) change by a multiplicative constant depending only on the dimension d. □

4. Model-free Itô integration

The key problem of "stochastic" integration with respect to typical price paths is, unsurprisingly, that they are not of bounded variation.

The model-free Itô integral presented in this section is a pathwise construction and comes with two natural interpretations in financial mathematics: in the case of existence the integral has a natural interpretation as the capital process of an adapted trading strategy and the set of paths where the integral does not exists allows for a model-free arbitrage opportunity of the first kind (cf. Proposition 2.5).

Roughly speaking, the construction of the model-free Itô integral is based on the existence of the quadratic variation for typical price paths and on an application of the pathwise Hoeffding inequality due to Vovk [Vov12] in the case of continuous price trajectories, and of the pathwise Burkholder-Davis-Gundy inequality due to Beiglböck and Siorpaes [BS15] in the case of price paths with jumps. Similarly as in classical stochastic integration, first we define the Itô integral for simple integrands and extend it via an approximation scheme to a larger class of integrands. Our simple integrands are the step functions:

A process $F\colon \Omega \times [0,T] \to \mathbb{R}^d$ is called *step function* if F is of the form

$$F_t := F_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} F_{\sigma_i} \mathbf{1}_{(\sigma_i, \sigma_{i+1}]}(t), \quad t \in [0,T], \tag{11}$$

where $(\sigma_i)_{i\in\mathbb{N}}$ is an increasing sequence of stopping times such that for each $\omega \in \Omega$ there exists an $N(\omega) \in \mathbb{N}$ with $\sigma_i(\omega) = \sigma_{i+1}(\omega)$ for all $i \geq N(\omega)$, $F_0 \in \mathbb{R}^d$ and $F_{\sigma_i}\colon \Omega \to \mathbb{R}^d$ is $\mathcal{F}_{\sigma_i}$-measurable. For such a step function F the corresponding integral process is well-defined for all $\omega \in \Omega$ and is given by

$$(F \cdot S)_t := \sum_{i=0}^{\infty} F_{\sigma_i} S_{\sigma_i \wedge t, \sigma_{i+1} \wedge t}, \quad t \in [0,T]. \tag{12}$$

Throughout the whole section we denote by $(\pi_n(\omega))_{n\in\mathbb{N}}$ the sequence of Lebesgue partitions consisting of the stopping times $(\tau_k^n(\omega))_{k\in\mathbb{N}}$ as introduced in Definition 3.10 for $\omega \in \Omega$, and the quadratic variation matrix of ω along $(\pi_n(\omega))_{n\in\mathbb{N}}$ is given by

$$[S]_t(\omega) := ([S^i, S^j]_t(\omega))_{1\leq i,j \leq d}, \quad t \in [0,T],$$

where we recall that $S_t^i(\omega) := \omega^i(t)$ for $\omega = (\omega^1, \dots, \omega^d)$ and we refer to Corollary 3.11 for the definition of $[S^i, S^j]_t(\omega)$. Recall that if the quadratic variation exists as a uniform limit, then it exists also in the sense of Föllmer along the same sequence of partitions (see [Vov15, Proposition 3]). Hence,

$$\int_0^t F_s^{\otimes 2} \,\mathrm{d}[S]_s := \sum_{i,j=1}^{d} \int_0^t F_s^i F_s^j \,\mathrm{d}[S^i, S^j]_s := \liminf_{n\to\infty} \sum_{k=0}^{\infty} \sum_{i,j=1}^{d} F_{\tau_k^n}^i F_{\tau_k^n}^j S^i_{\tau_k^n \wedge t, \tau_{k+1}^n \wedge t} S^j_{\tau_k^n \wedge t, \tau_{k+1}^n \wedge t},$$

$t \in [0,T]$, is actually a true limit for typical price paths.

Remark 4.1. *The existence of quadratic variation along the Lebesgue partitions is ensured for typical price paths belonging to the path spaces $\Omega \in \{\Omega_c, \Omega_+, \Omega_\psi\}$ under consideration. Indeed, in the case of Ω_+ we refer to Corollary 3.11 and the existence results in the case of Ω_c and Ω_ψ were established by Vovk (*[Vov12, Lemma 8.1] *and* [Vov15, Theorem 2]*).*

In the following we identify two functionals $X, Y\colon \Omega \times [0,T] \to \mathbb{R}^d$ if for typical price paths we have $X_t = Y_t$ for all $t \in [0,T]$, and we write $\overline{L}_0(\mathbb{R}^d)$ for the resulting space of equivalence classes which is equipped with the distance

$$d_\infty(X,Y) := \overline{E}[\|X-Y\|_\infty \wedge 1], \tag{13}$$

where $\|f\|_\infty := \sup_{t\in[0,T]}|f(t)|$ for $f\colon [0,T] \to \mathbb{R}^d$ denotes the supremum norm and $\overline{E}$ denotes an expectation operator defined for $Z\colon \Omega \to [0,\infty]$ by

$$\overline{E}[Z] := \inf\Big\{\lambda > 0 \,:\, \exists (H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda \text{ s.t. } \forall \omega \in \Omega \,:\, \liminf_{n\to\infty}(\lambda + (H^n \cdot S)_T(\omega)) \geq Z(\omega)\Big\}.$$

As in [PP16, Lemma 2.11] it can be shown that $(\overline{L}_0(\mathbb{R}^d), d_\infty)$ is a complete metric space and $(\overline{\mathcal{D}}(\mathbb{R}^d), d_\infty)$ is a closed subspace, where $\overline{\mathcal{D}}(\mathbb{R}^d)$ are those functionals in $\overline{L}_0(\mathbb{R}^d)$ which have a càdlàg representative.

The main result of this section is the following meta-theorem about model-free Itô integration.

Theorem 4.2. *Let $\Omega \in \{\Omega_c, \Omega_+, \Omega_\psi\}$. Then there exists a complete metric space $(\overline{H}, d_{\overline{H}})$ such that the (equivalence classes of) step functions are dense in $\overline{H}$ and such that the map*

$$F \mapsto (F \cdot S),$$

defined for step functions in (12)*, has a continuous extension that maps from $(\overline{H}, d_{\overline{H}})$ to $(\overline{\mathcal{D}}(\mathbb{R}), d_\infty)$. Moreover, $\overline{H}$ contains at least the càglàd adapted processes and if $(F_n) \subset \overline{H}$ is a sequence such that* $\sup_{\omega\in\Omega}\|F_n(\omega) - F(\omega)\|_\infty \to 0$*, then there exists a subsequence (F_{n_k}) with* $\lim_{k\to\infty}\|(F_{n_k} \cdot S)(\omega) - (F \cdot S)(\omega)\|_\infty = 0$ *for typical price paths ω.*

To prove this theorem, we will derive in the following subsections suitable continuity estimates for the integrals of step functions. Given these continuity estimates, the proof of Theorem 4.2 follows directly by approximating general integrands by step functions.

4.1. **Integration for continuous paths.** In this subsection we focus on the sample space $\Omega_c = C([0,T],\mathbb{R}^d)$ consisting of continuous paths $\omega\colon [0,T] \to \mathbb{R}^d$. The existence of the quadratic variation along the Lebesgue partitions for typical continuous price paths is ensured by [Vov12, Lemma 8.1] or [Vov15, Theorem 2].

We recover essentially the results of [PP16, Theorem 3.5] and are able to construct our integral for càglàd adapted integrands. However, in [PP16] we worked with the uniform topology on the space of integrands while here we are able to strengthen our results and to replace the uniform distance with a rather natural distance that depends only on the integral of the squared integrand against the quadratic variation. We are able to show that the closure of the step functions in this new distance contains the càglàd adapted processes. However, in principle this closure might contain a wider class of integrands.

The main ingredient in our construction is the following continuity estimate for the pathwise stochastic integral of a step function. It is based on Vovk's pathwise Hoeffding inequality.

Lemma 4.3 (Model-free concentration of measure, continuous version)**.** *Let $F\colon \Omega_c \times [0,T] \to \mathbb{R}^d$ be a step function. Then we have for all $a, b > 0$*

$$\overline{P}\Big(\{\|(F\cdot S)\|_\infty \geq a\sqrt{b}\} \cap \Big\{\int_0^T F_s^{\otimes 2}\,\mathrm{d}[S]_s \leq b\Big\}\Big) \leq 2\exp(-a^2/2).$$

Proof. Let $F_t = F_0 \mathbf{1}_0(t) + \sum_{m=0}^\infty F_m \mathbf{1}_{(\sigma_m,\sigma_{m+1}]}(t)$. For $n \in \mathbb{N}$ we define the stopping times

$$\zeta_0^n := 0, \qquad \zeta_{k+1}^n := \inf\{t \ge \zeta_k^n : |(F\cdot S)_{\zeta_k^n,t}|= 2^{-n}\},$$

and also $\tau_0^n := 0$, $\tau_{k+1}^n := \inf\{t \ge \tau_k^n : |S_{\tau_k^n,t}|= 2^{-n}\}$. We then write

$$\rho_0^n := 0, \qquad \rho_{k+1}^n := \inf\{t > \rho_k^n : t = \zeta_i^{2n} \text{ or } t = \tau_i^n \text{ or } t = \sigma_i \text{ for some } i \ge 0\}$$

for the union of the $(\zeta_k^{2n})_k$ and $(\tau_k^n)_k$ and $(\sigma_m)_m$. By definition of the times (ρ_k^n) we have

$$\sup_{t\in[0,T]} |(F\cdot S)_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t}| \le 2^{-2n}$$

and F is constant on $(\rho_k^n, \rho_{k+1}^n]$ for all k, and therefore Vovk's pathwise Hoeffding inequality, [Vov12, Theorem A.1] or [PP16, Lemma A.1], gives us for every $\lambda\in\mathbb{R}$ a strongly 1-admissible simple strategy $H^{\lambda,n} \in \mathcal{H}_1$ such that

$$1 + (H^{\lambda,n}\cdot S)_t \ge \exp\left(\lambda(F\cdot S)_t - \frac{\lambda^2}{2}\sum_{k=0}^\infty 2^{-4n}\mathbf{1}_{\{\rho_k^n\le t\}}\right) =: \mathcal{E}_t^{\lambda,n}, \qquad t\in[0,T]. \tag{14}$$

Next, observe that for all $i = 1,\dots,d$

$$\sup_{t\in[0,T]}\Big|\sum_{k=0}^\infty S^i_{\rho_k^n}\mathbf{1}_{[\rho_k^n,\rho_{k+1}^n)}(t) - S_t^i\Big| \le 2^{-n},$$

so since 2^{-n} decays faster than logarithmically, [PP16, Corollary 3.6] shows that for typical price paths we have for $i,j = 1,\dots,d$

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\Big|\sum_{k=0}^\infty S^i_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t} S^j_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t} - [S^i,S^j]_t\Big| = 0,$$

and using that $F(\omega)$ is piecewise constant for all $\omega\in\Omega_c$, we also get

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\Big|\sum_{k=0}^\infty F^i_{\rho_{k+}^n}F^j_{\rho_{k+}^n} S^i_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t} S^j_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t} - \int_0^t F_s^i F_s^j \,\mathrm{d}[S^i,S^j]_s\Big| = 0 \tag{15}$$

for typical price paths, where $F^i_{\rho_{k+}^n}$ is simply the value that F^i attains on $(\rho_k^n,\rho_{k+1}^n]$. We proceed by estimating for $k\ge 0$

$$2^{-2n} \le |(F\cdot S)_{\rho_k^n,\rho_{k+1}^n}| + 2^{-2n}\mathbf{1}_{\{\rho_k^n \text{ or } \rho_{k+1}^n=\tau_i^n \text{ or } \sigma_i \text{ for some } i\ge 0\}},$$

which together with (15) leads to

$$\begin{aligned}
\limsup_{n\to\infty}\sum_{k=0}^\infty 2^{-4n}\mathbf{1}_{\{\rho_k^n\le t\}} &\le \limsup_{n\to\infty}\Big(\sum_{k=0}^\infty |(F\cdot S)_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t}|^2 \\
&\qquad + 3\times 2^{-4n}\times(|\{k:\sigma_k\le t\}|+|\{k:\tau_k^n\le t\}|)\Big)\\
&= \limsup_{n\to\infty}\Big(\sum_{k=0}^\infty\sum_{i,j=1}^d F^i_{\rho_{k+}^n}F^j_{\rho_{k+}^n} S^i_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t} S^j_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t}\\
&\qquad + 3\times 2^{-4n}\times|\{k:\tau_k^n\le t\}|\Big)\\
&= \int_0^t F_s^{\otimes 2}\,\mathrm{d}[S]_s + \limsup_{n\to\infty} 3\times 2^{-4n}\times|\{k:\tau_k^n\le t\}|
\end{aligned}$$

for typical price paths. For typical price paths we also have $\lim_{n\to\infty} 2^{-2n} \times |\{k : \tau_k^n \le t\} = \sum_{i=1}^d [S^i, S^i]_t$, and consequently

$$\limsup_{n\to\infty} \sum_{k=0}^{\infty} 2^{-4n} \mathbf{1}_{\{\rho_k^n \le t\}} \le \int_0^t F_s^{\otimes 2} \,\mathrm{d}[S]_s, \quad t \in [0,T]. \tag{16}$$

Plugging (16) into (14), we get for typical price paths on the set $\{\|(F\cdot S)\|_\infty \ge a\sqrt{b}\} \cap \{\int_0^T F_s^{\otimes 2}\,\mathrm{d}[S]_s \le b\}$ that

$$\liminf_{n\to\infty} \sup_{t\in[0,T]} \frac{\mathcal{E}_t^{\lambda,n} + \mathcal{E}_t^{-\lambda,n}}{2} \ge \frac{1}{2}\exp\left(\lambda a\sqrt{b} - \frac{\lambda^2}{2} b\right).$$

Taking $\lambda = a/\sqrt{b}$, the right hand side becomes $1/2\exp(a^2/2)$. Our claim then follows from [PP16, Remark 2.2] which states that it suffices to superhedge with the time-supremum rather than at the terminal time. $\square$

Remark 4.4. *No part of the proof was based on the fact that we are in a finite-dimensional setting, and the same arguments extend without problems to the case where we have a countable number of assets $(S^i)_{i\in\mathbb{N}}$ or even an uncountable number $(S^i)_{i\in I}$ but an integrand F with $F^i \ne 0$ only for countably many $i \in I$ (which is needed to have $\int_0^\cdot F_s^{\otimes 2}\,\mathrm{d}[S]_s$ well-defined).*

Our next aim is to extend the stochastic integral from step functions to more general integrands. For that purpose we define

$$H^2 := \Big\{F\colon \Omega_c \times [0,T] \to \mathbb{R}^d \,:\, \int_0^T F_s^{\otimes 2}\,\mathrm{d}[S]_s < \infty \text{ for typical price paths}\Big\},$$

we identify $F, G \in H^2$ if $\int_0^T (F_s - G_s)^{\otimes 2}\,\mathrm{d}[S]_s = 0$ for typical price paths, and we write $\overline{H}^2$ for the space of equivalence classes, which we equip with the distance

$$d_{\mathrm{QV}}(F,G) := \overline{E}\Big[\int_0^T (F_t - G_t)^{\otimes 2}\,\mathrm{d}[S]_t \wedge 1\Big]. \tag{17}$$

Arguing as in [PP16, Lemma 2.11] it is straightforward to show that $(\overline{H}^2, d_{\mathrm{QV}})$ is a complete metric space. If now F and G are step functions, for any $\varepsilon, \delta > 0$ we obtain from Lemma 4.3 the following estimate

$$\begin{aligned} d_\infty((F\cdot S),(G\cdot S)) &\le \overline{P}(\|((F-G)\cdot S)\|_\infty \ge \varepsilon) + \varepsilon \\ &\le \overline{P}\Big(\{\|((F-G)\cdot S)\|_\infty \ge \varepsilon\} \cap \Big\{\int_0^T (F_t - G_t)^2\,\mathrm{d}[S]_t \le \delta\Big\}\Big) \\ &\quad + \overline{P}\Big(\int_0^T (F_t - G_t)^2\,\mathrm{d}[S]_t > \delta\Big) + \varepsilon \\ &\le 2\exp\Big(-\frac{\varepsilon^2}{2\delta}\Big) + \frac{d_{\mathrm{QV}}(F,G)}{\delta} + \varepsilon. \end{aligned}$$

Hence, setting $\delta := d_{\mathrm{QV}}(F,G)^{2/3}$ and $\varepsilon := \sqrt{\delta|\log\delta|}$ we get

$$d_\infty((F\cdot S),(H\cdot S)) \le 2\delta^{1/2} + d_{\mathrm{QV}}(F,G)^{1/3} + \sqrt{\delta|\log\delta|} \lesssim d_{\mathrm{QV}}(F,G)^{1/3}|\log(d_{\mathrm{QV}}(F,G))|.$$

So we can extend the stochastic integral to the closure of the step functions in $(\overline{H}^2, d_{\mathrm{QV}})$. Since we do not understand this closure very well we introduce a localized version of d_{QV}, as in [PP16]: For $c > 0$ we define

$$d_{\mathrm{QV},c}(F,G) := \overline{E}\bigg[\bigg(\int_0^T (F_t - G_t)^{\otimes 2}\,\mathrm{d}[S]_t \wedge 1\bigg)\mathbf{1}_{\{|[S]|_T \le c\}}\bigg], \tag{18}$$

where we wrote $|[S]_T| := \big(\sum_{i,j=1}^d [S^i, S^j]_T^2\big)^{1/2}$. We also set

$$d_{\infty,c}(F,G) := \overline{E}[(\|F-G\|_\infty \wedge 1)\mathbf{1}_{\{|[S]|_T \le c\}}]. \tag{19}$$

Then the same computation as before shows that

$$d_{\infty,c}((F\cdot S),(H\cdot S)) \lesssim d_{\mathrm{QV},c}(F,G)^{1/3}|\log(d_{\mathrm{QV},c}(F,G))|$$

for all step functions F and G and all $c > 0$. From here the same approximation scheme as in [PP16, Theorem 3.5] shows that the closure of the step functions in the metric $d_{\mathrm{QV},loc} := \sum_n 2^{-n}(d_{\mathrm{QV},n} \wedge 1)$ contains at least the càglàd adapted processes.

4.2. **Integration for non-negative càdlàg paths.** The construction of model-free Itô integrals with respect to càdlàg price paths requires different techniques compared to those used in Subsection 4.1 for continuous price paths. While there, using the Lebesgue stopping times, we had a very precise control of the fluctuations of continuous price paths, this is not possible anymore in the presence of jumps as now price paths could have a "big" jump at any time. In particular, this prevents us from applying Vovk's pathwise Hoeffding inequality.

Here we consider the sample space $\Omega_+ = D_{S_0}([0,T],\mathbb{R}^d)$ of non-negative càdlàg paths starting from $S_0 \in \mathbb{R}^d_+$. Based on the pathwise Burkholder-Davis-Gundy inequality due to Beiglböck and Siorpaes [BS15], we obtain the following model-free bound on the magnitude of the pathwise stochastic integral.

Lemma 4.5 (Integral estimate, non-negative càdlàg version). *For a step function $F\colon \Omega_+ \times [0,T] \to \mathbb{R}^d$ and for $a,b,c > 0$ one has*

$$\overline{Q}\bigg(\{\|(F\cdot S)\|_\infty \ge a\} \cap \bigg\{\int_0^T F_t^{\otimes 2}\,\mathrm{d}[S]_t \le b\bigg\} \cap \{\|F\|_\infty \le c\}\bigg) \le \frac{6\sqrt{b}+2c}{a},$$

where $\overline{Q}$ denotes the set function of Definition 2.6.

As a foundation for the proof let us briefly recall the pathwise version of the Burkholder-Davis-Gundy inequality [BS15, Theorem 2.1]: If $n \in \mathbb{N}$, $k = 0,\dots,n$, $x_k \in \mathbb{R}$ and $x_k^* := \max_{0\le l\le k}|x_l|$, then

$$\max_{0\le k\le n}|x_k| \le 6\bigg(|x_0|^2 + \sum_{k=0}^{n-1}|x_{k+1}-x_k|^2\bigg)^{\frac{1}{2}} + 2(h\cdot x)_n, \tag{20}$$

where

$$(h\cdot x)_n := \sum_{k=0}^{n-1} h_k(x_{k+1}-x_k) \quad \text{with} \quad h_k := \frac{x_k}{\sqrt{[x]_k + ((x)_k^*)^2}} \tag{21}$$

and with the convention $\frac{0}{0} = 0$. With this purely deterministic inequality at hand we are ready to prove Lemma 4.5.

Proof of Lemma 4.5. Let $F\colon \Omega_+ \times [0,T] \to \mathbb{R}^d$ be a step function of the form (11), i.e.

$$F_t := F_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{\infty} F_{\sigma_i} \mathbf{1}_{(\sigma_i,\sigma_{i+1}]}(t), \quad t \in [0,T],$$

for some sequence of stopping times $(\sigma_i)_{i\in\mathbb{N}}$. For $n \in \mathbb{N}$ we recall that $(\tau_j^n)_{j\in\mathbb{N}}$ is the sequence of Lebesgue stopping times defined in Definition 3.10 and denote by $(\rho_k^n)_{k\in\mathbb{N}}$ the union of $(\sigma_i)_{i\in\mathbb{N}}$ and $(\tau_j^n)_{j\in\mathbb{N}}$ with redundancies deleted. Setting $\tilde{\rho}_k^n := \max\{\sigma_l^n \in (\sigma_l^n)_{l\in\mathbb{N}} : \sigma_l^n \le \tau_k^n\}$ for $k \in \mathbb{N}$, it is straightforward to see that

$$F_t = F_t^n := F_0 \mathbf{1}_{\{0\}}(t) + \sum_{k=0}^{\infty} F_{\tilde{\rho}_k^n} \mathbf{1}_{(\rho_k^n,\rho_{k+1}^n]}(t), \quad t \in [0,T],$$

and thus

$$(F\cdot S)_t = \sum_{i=0}^{\infty} F_{\sigma_i} S_{\sigma_i\wedge t,\sigma_{i+1}\wedge t} = \sum_{k=0}^{\infty} F_{\tilde{\rho}_k^n} S_{\rho_k^n\wedge t,\rho_{k+1}^n\wedge t} = (F^n\cdot S)_t, \quad t \in [0,T].$$

In order to apply the pathwise Burkholder-Davis-Gundy inequality (20), we define iteratively $x_0^n := 0$ and

$$x_{k+1}^n := x_k^n + F_{\tilde{\rho}_k} S_{\rho_k^n\wedge T,\rho_{k+1}^n\wedge T}, \quad k \in \mathbb{N},$$

and therefore (20) yields

$$\sup_{k\in\mathbb{N}} |x_k^n| = \sup_{k\in\mathbb{N}} |(F^n\cdot S)_{\rho_k^n\wedge T}| \le 6\bigg(\sum_{k=0}^{\infty} (F_{\tilde{\rho}_k^n} S_{\rho_k^n\wedge T,\rho_{k+1}^n\wedge T})^2\bigg)^{\frac{1}{2}} + 2(h^n\cdot x)_T.$$

Due to the definition of the Lebesgue stopping times $(\tau_j^n)_{j\in\mathbb{N}}$, this leads to the continuous time estimate

$$\sup_{t\in[0,T]} |(F^n\cdot S)_t| \le 6\bigg(\sum_{k=0}^{\infty}\sum_{i,j=1}^{d} F^i_{\tilde{\rho}_k^n} F^j_{\tilde{\rho}_k^n} S^i_{\rho_k^n\wedge T,\rho_{k+1}^n\wedge T} S^j_{\rho_k^n\wedge T,\rho_{k+1}^n\wedge T}\bigg)^{\frac{1}{2}} + 2(\phi^n\cdot S)_T + \|F\|_\infty 2^{-n}$$

where $F = (F^1,\dots,F^d)$ and ϕ^n is the adapted simple trading strategy given by the position $\phi_k^n := h_k^n F_{\tilde{\rho}_k^n}$ with h_k^n defined as in (21). To turn ϕ^n into a weakly admissible strategy, we introduce the stopping time

$$\vartheta^n := \inf\bigg\{t \ge 0 : \sum_{k=0}^{\infty}\sum_{i,j=1}^{d} F^i_{\tilde{\rho}_k^n} F^j_{\tilde{\rho}_k^n} S^i_{\rho_k^n\wedge T,\rho_{k+1}^n\wedge T} S^j_{\rho_k^n\wedge T,\rho_{k+1}^n\wedge T} > b\bigg\} \wedge \inf\{t\ge 0 : |F_t| \ge c\} \wedge T$$

for $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned}\sup_{t\in[0,\vartheta^n]} &|(F^n\cdot S)_t| \\ &\le 6\bigg(\sum_{k=0}^{\infty}\sum_{i,j=1}^{d} F^i_{\tilde{\rho}_k^n} F^j_{\tilde{\rho}_k^n} S^i_{\rho_k^n\wedge\vartheta^n,\rho_{k+1}^n\wedge\vartheta^n} S^j_{\rho_k^n\wedge\vartheta^n,\rho_{k+1}^n\wedge\vartheta^n}\bigg)^{\frac{1}{2}} + 2(\mathbf{1}_{[0,\vartheta^n]}\phi^n\cdot S)_T + c2^{-n}\end{aligned}$$

and in particular $\mathbf{1}_{[0,\vartheta^n]} 2\phi^n$ is weakly $(6\sqrt{b}+2^{-n}c+2c)$-admissible since $|\phi^n| \le \|F\|_\infty$ for every $n \in \mathbb{N}$. Taking the limit inferior as $n\to\infty$, one gets

$$\liminf_{n\to\infty} \sup_{t\in[0,\vartheta^n]} |(F\cdot S)_t| \le 6\bigg(\int_0^T F_t^{\otimes 2}\,\mathrm{d}[S]_t\bigg)^{\frac{1}{2}} + 2\liminf_{n\to\infty} (\mathbf{1}_{[0,\vartheta^n]}\phi^n\cdot S)_T.$$

Hence, we deduce

$$\overline{Q}\bigg(\{\|(F\cdot S)\|_\infty \geq a\} \cap \bigg\{\int_0^T F_t^{\otimes 2}\,\mathrm{d}[S]_t \leq b\bigg\} \cap \{\|F\|_\infty \leq c\}\bigg) \leq \frac{6\sqrt{b}+2c}{a}.$$

□

Corollary 4.6. *For $a,b,c>0$ and any step function $F\colon\Omega_+\to\mathbb{R}^d$ one has*

$$\overline{P}(\{\|(F\cdot S)\|_\infty \geq a\} \cap \{\|F\|_\infty \leq c\} \cap \{|[S]_T| \leq b\}) \leq (1+\mathbb{1}\cdot S_0)\frac{(6\sqrt{b}+2)c}{a},$$

where we recall that $\mathbb{1}=(1,\dots,1)\in\mathbb{R}^d$ and $|[S]_T|=\big(\sum_{i,j=1}^d [S^i,S^j]_T^2\big)^{1/2}$.

Proof. Using the monotonicity of $\overline{P}$, Lemma 2.8 and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\overline{P}(\{\|(F\cdot S)\|_\infty \geq a\} \cap \{\|F\|_\infty \leq c\} \cap \{|[S]_T| \leq b\})\\
&\qquad \leq \overline{P}\bigg(\{\|(F\cdot S)\|_\infty \geq a\} \cap \bigg\{\int_0^T F_t^{\otimes 2}\,\mathrm{d}[S]_t \leq bc^2\bigg\} \cap \{\|F\|_\infty \leq c\}\bigg)\\
&\qquad \leq (1+\mathbb{1}\cdot S_0)\overline{Q}\bigg(\{\|(F\cdot S)\|_\infty \geq a\} \cap \bigg\{\int_0^T F_t^{\otimes 2}\,\mathrm{d}[S]_t \leq bc^2\bigg\} \cap \{\|F\|_\infty \leq c\}\bigg).
\end{aligned}$$

Combing this estimate with Lemma 4.5 gives the assertion. □

As in (13) and (19) we introduce the (pseudo-)distances d_∞ and $d_{\infty,c}$ on the (equivalence classes) of processes from $\Omega_+\times[0,T]$ to $\mathbb{R}^d$. For two step functions F and G and $a,b,c>0$ Corollary 4.6 gives

$$\begin{aligned}
d_{\infty,b}((F\cdot S),(G\cdot S)) &\leq \overline{P}(\{\|((F-G)\cdot S)\|_\infty \geq a\} \cap \{|[S]_T| \leq b\}) + a\\
&\leq \overline{P}(\{\|((F-G)\cdot S)\|_\infty \geq a\} \cap \{\|F-G\|_\infty \leq c\} \cap \{|[S]_T| \leq b\})\\
&\quad + \overline{P}(\{\|F-G\|_\infty \geq c\} \cap \{|[S]_T| \leq b\}) + a\\
&\leq (1+\mathbb{1}\cdot S_0)\frac{(6\sqrt{b}+2)c}{a} + \frac{1}{c}d_{\infty,b}(F,G) + a.
\end{aligned}$$

Setting $a := d_{\infty,b}(F,G)^{\frac{1}{3}}$ and $c := d_{\infty,b}(F,G)^{\frac{2}{3}}$ leads to

$$d_{\infty,b}((F\cdot S),(G\cdot S)) \leq (1+\mathbb{1}\cdot S_0)(6\sqrt{b}+4)d_{\infty,b}(F,G)^{\frac{1}{3}}. \tag{22}$$

Based on this observation we can use the same ideas as in [PP16, pp. 14-15] to construct the model-free Itô integral of càglàd integrands with respect to non-negative price paths.

4.3. **Integration for paths with mildly restricted jumps.** In this subsection we consider the space Ω_ψ of càdlàg paths with mildly restricted jumps, which completes the picture of Itô integration with respect to typical price paths. For this purpose we again establish a model-free bound for the integrals against step functions, which can be seen as the counterpart to Lemma 4.5 and its proof also replies on the pathwise Burkholder-Davis-Gundy inequality due to Beiglböck and Siorpaes [BS15].

Lemma 4.7 (Integral estimate, càdlàg version for mildly restricted jumps)**.** *Let $F\colon \Omega_\psi \times [0,T] \to \mathbb{R}^d$ be a step function. For $a,b,m,M>0$ the following estimate holds*

$$\overline{P}\bigg(\{\|(F\cdot S)\|_\infty \geq a\} \cap \bigg\{\int_0^T F_t^{\otimes 2}\,\mathrm{d}[S]_t \leq b\bigg\} \cap \{\|F\|_\infty \leq m\} \cap \{\|S\|_\infty \leq M\}\bigg) \leq \frac{6}{a}\sqrt{2b + 2m^2\cdot\psi(M)^2}.$$

Proof. The proof goes along the same lines as the proof of Lemma 4.5 and thus we keep it very short. Let $F\colon \Omega_\psi \times [0,T] \to \mathbb{R}^d$ be a step function of the form (11). For $n \in \mathbb{N}$ we recall that $(\tau_j^n)_{j\in\mathbb{N}}$ is the sequence of Lebesgue stopping times as defined in Definition 3.10 and denote by $(\rho_k^n)_{k\in\mathbb{N}}$ the union of $(\sigma_i)_{i\in\mathbb{N}}$ and $(\tau_j^n)_{j\in\mathbb{N}}$ with redundancies deleted. Setting $\tilde{\rho}_k^n := \max\{\sigma_l^n \in (\sigma_l^n)_{l\in\mathbb{N}} : \sigma_l^n \leq \tau_k^n\}$ for $k \in \mathbb{N}$, it is straightforward to see that

$$F_t = F_t^n := F_0 \mathbf{1}_0(t) + \sum_{k=0}^\infty F_{\tilde{\rho}_k^n} \mathbf{1}_{(\rho_k^n, \rho_{k+1}^n]}(t), \quad t \in [0,T].$$

Now let us define the following stopping time

$$\tau_{b,m,M}^n := \inf\bigg\{t \geq 0 : \sum_{j=0}^\infty (F_{\tilde{\rho}_j^n} S_{\rho_j^n\wedge t, \rho_{j+1}^n \wedge t})^2 \geq b \text{ or } |S_t| \geq M \text{ or } |F_t| \geq m\bigg\} \wedge T.$$

Using the pathwise Burkholder-Davis-Gundy inequality (20) we get

$$\sup_{t\in[0,\tau_{b,m,M}^n]} |(F^n\cdot S)_t| \leq 6\bigg(\sum_{j=0}^\infty (F_{\tilde{\rho}_k^n} S_{\rho_j^n \wedge \tau_{b,m,M}^n \rho_{j+1}^n \wedge \tau_{b,m,M}^n})^2\bigg)^{1/2} + 2(\phi^n \mathbf{1}_{[0,\tau_{b,M}^n]}\cdot S)_T$$

for some sequence of simple trading strategies $(\phi^n)_{n\in\mathbb{N}}$. From the definition of $\tau_{b,m,M}^n$ and the fact that $\Delta S_{\tau_{b,m,M}^n} \leq \psi(M)$ we see that the strategy $2\phi^n \mathbf{1}_{[0,\tau_{b,m,M}^n]}$ is $6(2b + 2m^2\psi(M)^2)^{1/2}$-admissible. Proceeding to the limit, similarly as in the proof of Lemma 4.5, completes the proof. □

Corollary 4.8. *For $a,b,m,M>0$ and any step function $F\colon \Omega_\psi \times [0,T] \to \mathbb{R}^d$ one has*

$$\overline{P}(\{\|(F\cdot S)\|_\infty \geq a\} \cap \{|[S]_T| \leq b\} \cap \{\|F\|_\infty \leq m\} \cap \{\|S\|_\infty \leq M\}) \leq \frac{6m}{a}\sqrt{2b + 2\psi(M)^2},$$

where we recall that $|[S]_T| := \big(\sum_{i,j=1}^d [S^i, S^j]_T^2\big)^{1/2}$.

As in (13) and (19) we introduce the (pseudo-)distance d_∞ on the space (of equivalence classes) of adapted processes from $\Omega_\psi \times [0,T]$ to $\mathbb{R}^d$ and set

$$d_{c,M}(X,Y) := \overline{E}[(\|X-Y\|_\infty \wedge 1)\mathbf{1}_{\Omega_{c,M}}]$$

for $c, M > 0$ and

$$\Omega_{c,M} := \{|[S]_T| \leq c \text{ and } \|S\|_\infty \leq M\}.$$

From Corollary 4.8 we get

$$\begin{aligned} d_{c,M}((F\cdot S),(G\cdot S)) &\leq \overline{P}(\{\|((F-G)\cdot S)\|_\infty \geq a\}\cap \Omega_{c,M}) + a \\ &\leq \overline{P}(\{\|((F-G)\cdot S)\|_\infty \geq a\}\cap \{\|F-G\|_\infty \geq m\}\cap \Omega_{c,M}) \\ &\quad + \overline{P}(\{\|((F-G)\cdot S)\geq a\|_\infty\}\cap \{\|F-G\|_\infty \leq m\}\cap \Omega_{c,M}) + a \\ &\leq \frac{6m}{a}\sqrt{2c+2\psi(M)^2} + \frac{d_{c,M}(F,G)}{m} + a \end{aligned}$$

for step functions F, G and $a, c, m, M > 0$. Setting

$$a := d_{c,M}(F,G)^{1/3} \quad \text{and} \quad m := d_{c,M}(F,G)^{2/3},$$

we deduce that

$$d_{c,M}((F\cdot S),(G\cdot S)) \leq (6\sqrt{2c+2\psi(M)^2}+2)d_{c,M}(F,G)^{1/3}. \tag{23}$$

Based on this observation we can again extent the construction of the model-free Itô integral from simple integrands to càglàd integrands with respect to typical càdlàg price paths with mildly restricted jumps. Notice, that we cannot simply take $c = M$ for the extension since ψ could grow arbitrarily fast.

Remark 4.9. *In the very recent work* [Vov16] *Vovk introduces a slightly generalized version of the space* Ω_ψ *which allows to have two independent predicable bounds for the jumps of possible price paths, i.e. a bound for jumps going upwards and a second bound for jumps going downwards. In his note, Vovk also obtains the convergence of non-anticipating Riemann sums along a fixed sequence of partitions, which thus results in a model-free Itô integral. However, this construction currently provides only a preliminary version of "stochastic" integration in a model-free stetting since it might depend on the sequence of partitions and no continuity estimates are given. Furthermore, it works with different techniques compared to the present work.*

Appendix A. Properties of Vovk's outer measure

This appendix collects postponed proofs from the previous sections and an elementary result (Borel-Cantelli lemma) which was used for the construction of the quadratic variation and the model-free Itô integrals.

Proof of Proposition 2.4. Let $\lambda > 0$ and let $(H^n)_{n\in\mathbb{N}} \subseteq \mathcal{H}_\lambda$ be such that $\liminf_n(\lambda + (H^n\cdot S)_T) \geq \mathbf{1}_A$. Then

$$\mathbb{P}(A) \leq \mathbb{E}_\mathbb{P}[\liminf_n(\lambda + (H^n\cdot S)_T)] \leq \liminf_n \mathbb{E}_\mathbb{P}[\lambda + (H^n\cdot S)_T] \leq \lambda,$$

where in the last step we used that $\lambda + (H^n\cdot S)$ is a non-negative càdlàg $\mathbb{P}$-local martingale with $\mathbb{E}_\mathbb{P}[|\lambda + (H^n\cdot S)_0|] < \infty$ and thus a $\mathbb{P}$-supermartingale. □

Proof of Proposition 2.5. If $\overline{P}(A) = 0$, then for every $n \in \mathbb{N}$ there exists a sequence of simple strategies $(H^{n,m})_{m\in\mathbb{N}} \subset \mathcal{H}_{2^{-n-1}}$ such that $2^{-n-1} + \liminf_{m\to\infty}(H^{n,m}\cdot S)_T(\omega) \geq \mathbf{1}_A(\omega)$ for all $\omega \in \Omega$. For $K \in (0,\infty)$ set $G^m := K\sum_{n=0}^m H^{n,m}$, and thus $G^m \in \mathcal{H}_K$. For every $k \in \mathbb{N}$ one gets

$$\liminf_{m\to\infty}(K + (G^m\cdot S)_T) \geq \sum_{n=0}^k (2^{-n-1}K + \liminf_{m\to\infty}(H^{n,m}\cdot S)_T) \geq (k+1)K\mathbf{1}_A.$$

Because the left hand side does not depend on k, the sequence (G^m) satisfies (1).

Conversely, if there exist a constant $K \in (0,\infty)$ and a sequence of K-admissible simple strategies $(H^n)_{n\in\mathbb{N}} \subset \mathcal{H}_K$ satisfying (1), then we can scale it down by an arbitrary factor $\varepsilon > 0$ to obtain a sequence of strategies in $\mathcal{H}_\varepsilon$ that superhedge $\mathbf{1}_A$, which implies $\overline{P}(A) = 0$. □

As the proof of the classical Borel-Cantelli lemma requires only countable subadditivity, Vovk's outer measure allows for a version of the Borel-Cantelli lemma.

Lemma A.1. *Let $(A_j)_{j\in\mathbb{N}} \subset \Omega$ be a sequence of events. If $\sum_{j=1}^\infty \overline{P}(A_j) < \infty$, then*

$$\overline{P}\Big(\bigcap_{i=1}^\infty \bigcup_{j=i}^\infty A_j\Big) \le \liminf_{i\to\infty} \overline{P}\Big(\bigcup_{j=i}^\infty A_j\Big) \le \liminf_{i\to\infty} \sum_{j=i}^\infty \overline{P}(A_j) = 0.$$

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