# Kissing numbers: Surprises in Dimension Four

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The "kissing number problem" is a basic geometric problem that got its name from billards: Two balls "kiss" if they touch. The kissing number problem asks how many other balls can touch one given ball at the same time. If you arrange the balls on a pool table, it is easy to see that the answer is exactly six: Six balls just perfectly surround a given ball.
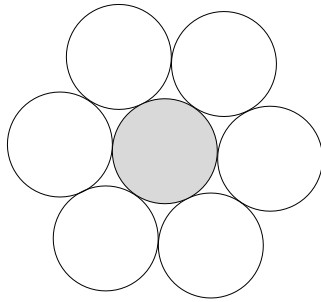
If, however, you think about this as a three-dimensional problem, the question "how many balls can touch a given ball at the same time" becomes much more interesting — and quite complicated. In fact, The 17th century scientific genius Sir Isaac Newton and his colleague David Gregory had a controversy about this in 1694 — Newton said that 12 should be the correct answer, while Gregory thought that 13 balls could be fitted so that they would touch a given ball (all of them of the same size) simultaneously. One of the famous platonic solids, the icosahedron, in fact yields a configuration of 12 touching balls that has great beauty and symmetry — and leaves considerable gaps between the touching balls (top right). So perhaps if you move all of them to one side, would a 13th ball possibly fit in? The answer is no — 12 is the correct answer — but to *prove this* is a hard problem, which was finally solved by Schütte and van der Waerden in 1953.

Mathematicians worry about the same problem also for higher dimensional spheres, say for four-dimensional balls in four-dimensional space. Why should they? Well, one answer is that good sphere packings are closely related to good error-correcting codes, and thus the geometry of sphere packings in high dimensional spaces is important for the mathematical theory of error-correcting codes, and thus to one of the mathematical core technologies that play there perfect role in every day life (without us usually noticing that).

So the kissing number problem may be posed in n-dimensional space, and in fact the mathematical theory to describe sphere packings is just basic linear algebra, as one learns it in college. The mathematical theory that allows one to sometimes *solve* such problems is much more complicated: Philippe Delsarte (Phillips Research Labs) in 1973 described a "linear programming method" that does allow one to prove good bounds on the maximal number of balls that would kiss a given one in n-dimensional space. Spectacular breakthroughs on this problem occurred in the late seventies, when Andrew Odlyzko and Neil Sloane (at AT&T Bell Labs)
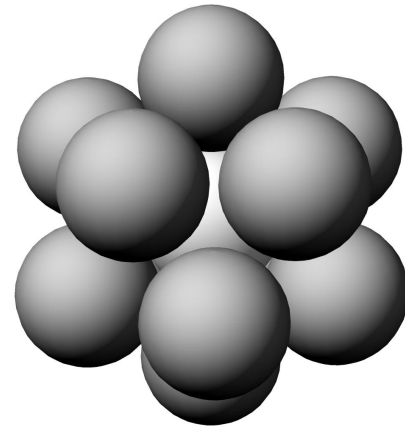
Graphic: Detlev Stalling, ZIB Berlin

and at the same time Vladimir I. Levenstein in Russia proved that the correct, exact maximal numbers for the kissing number problem are 240 in dimension 8, and 196560 in dimension 24. This is amazing, because these are also the only two dimensions where one knows a precise answer. It depends on the fact that mathematicians know very remarkable configurations in dimensions 8 and 24, which they call the E_8 *lattice* and the *Leech lattice*, respectively.

So the kissing number problem remained unsolved, in particular, for the case of dimension four. The so-called *24-cell*, a four-dimensional "platonic solid" of remarkable beauty (next page), yields a configuration of 24 balls that would touch a given one in four-dimensional space. But is 24 the answer? It was *proved* that with Delsarte's method the best upper bound one could get is 25. So it comes as a great surprise that now the Russian mathematician Oleg Musin, who lives in Los Angeles, has indeed found a method to modify Delsarte's method in a very beautiful and clever way, and thus improve the upper bound from 25 to 24.

So indeed: 24 is the answer!

Oleg Musin

Surprisingly, this is not the only spectacular recent piece of progress related to the packing of spheres in high-dimensional space. Namely, by again extending and improving upon Delsarte's method, Henry Cohn (Microsoft Research) in joint works with Noam Elkies (Harvard University) and with Abhinav Kumar (a mathematics graduate student at Harvard) has obtained new upper bounds on the density of packings of balls of equal size in

n-dimensional space, that is, on the fraction of space that can be filled by packing balls of equal size. Indeed, for dimensions 8 and 24 they obtained upper bounds that are off by a fraction of one part in 10^{27} from the best known lattice sphere packings — which are given by the E_8 lattice and the Leech lattice. Both bounds are amazingly close, and there still is hope to prove that, indeed, the E_8 lattice and the Leech lattice are the optimal ways to pack billard balls in 8-dimensional respectively 24-dimensional space. This would be quite amazing: Just recall that the Kepler conjecture about the packing of equal balls in three-dimensional space was only recently resolved (by Thomas C. Hales), the proofs were very difficult and used a lot of computer calculations, there was lots of controversy, and still no proof is published! It seems that geometry in dimensions 4, 8 or 24 has many surprises — and may be easier and provide nicer answers than geometry in three-dimensional space (at least to some mathematicians).

Oleg Musin and Henry Cohn presented their work to an international audience of experts at the workshop "Combinatorial and Discrete Geometry" (November 17 to 21, 2003), at MSRI.
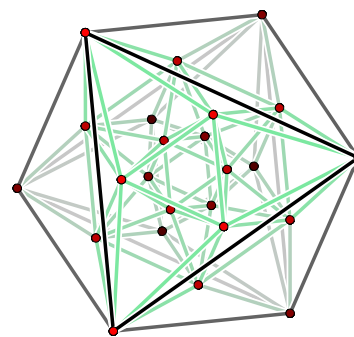
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Forthcoming Workshops

Most of these workshops are offered under the auspices of one of the current programs (see Director's Notes starting on page 1). For more information about the programs and workshops, see http://www.msri.org/calendar.

January 12 to January 16, 2004: *Introductory Workshop in Topological Aspects of Real Algebraic Geometry*, organized by Selman Akbulut, Grisha Mikhalkin, Victoria Powers, Boris Shapiro, Frank Sottile, and Oleg Viro.

February 9 to February 13, 2004: *Genetics of Complex Disease*, organized by Jun Liu, Mary Sara McPeek, Richard Olshen (chair), David O. Siegmund, and Wing Wong.

February 23 to February 27, 2004: *Topology and Geometry of Real Algebraic Varieties*, organized by Viatcheslav Kharlamov, Boris Shapiro, and Oleg Viro.

March 15 to March 19, 2004: *Mathematical Neuroscience*, organized by Paul C. Bressloff, Jack D. Cowan (chair), G. Bard Ermentrout, Mary Pugh, and Terry J. Sejnowski.

March 22 to March 26, 2004:: *Symplectic Geometry and Mathematical Physics*, organized by Denis Auroux, Dan Freed, Helmut Hofer, Francis Kirwan, and Gang Tian.

April 12 to April 16, 2004: *Algorithmic, Combinatorial and Applicable Real Algebraic Geometry*, organized by Lalo Gonzalez-Vega, Victoria Powers, and Frank Sottile.

June 14 to June 18, 2004: *Analysis of Algorithms*, organized by P. Flajolet, P. Jacquet, H. Prodinger, G. Seroussi, R. Sedgewick, W. Szpankowski, B. Vallée, and M. Weinberger.

June 22 to June 25, 2004: *Tenth Annual Conference for African American Researchers in the Mathematical Sciences*.

Current and Recent Workshops

Most recent first. For information see http://www.msri.org/calendar.

December 1 to December 5, 2003: *Geometric Analysis*, organized by Ben Chow, Peter Li, Richard Schoen (chair), and Richard Wentworth.

November 17 to November 21, 2003: *Combinatorial and Discrete Geometry*, organized by Jesús A. De Loera, Jacob E. Goodman, János Pach and Günter M. Ziegler.

November 9 to November 13, 2003: *Floer homology for 3-manifolds*, organized by Yasha Eliashberg, Robion Kirby and Peter Kronheimer.

October 13 to October 17, 2003: *Mathematical Foundations of Geometric Algorithms*, organized by Pankaj Agarwal, Herbert Edelsbrunner, Micha Sharir, and Emo Welzl.

September 27 to September 28, 2003: *Texas Southern University/MSRI Workshop on Modern Mathematics: An Introduction to 2004–05 Programs at the Mathematical Sciences Research Institute*, organized by Nathaniel Dean and Robert Megginson.