A flag vector of a 3-sphere that is not the flag vector of a 4-polytope*

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Abstract

We present a first example of a flag vector of a polyhedral sphere that is not the flag vector of any polytope. Namely, there is a unique 3-sphere with the parameters $(f_0, f_1, f_2, f_3; f_{02}) = (12, 40, 40, 12; 120)$, but this sphere is not realizable by a convex 4-polytope. The 3-sphere, which is 2-simple and 2-simplicial, was found by Werner (2009); we present results of a computer enumeration which imply that the sphere with these parameters is unique. We prove that it has no oriented matroid, and thus is not realizable; this proof was also found by computer, but can be verified by hand.

1 Introduction

A lot of work has gone into the characterization of the sets of all f-vectors $(f_0, f_1, \ldots, f_{d-1})$ of convex d-polytopes, which we denote by $f(\mathcal{P}^d)$. Already in 1906, Steinitz [23] characterized the f-vectors of 3-polytopes as

$$f(\mathcal{P}^3) = \{(f_0, f_1, f_2) \in \mathbb{Z}^3 : f_0 - f_1 + f_2 = 2,\ f_2 \leq 2f_0 - 4,\ f_0 \leq 2f_2 - 4\}.$$

Grünbaum [13, p. 131] showed that the affine hull of $f(\mathcal{P}^d)$ is a hyperplane in $\mathbb{R}^d$, determined by the Euler equation. Moreover, the f-vectors of the simplicial polytopes, which we denote by $f(\mathcal{P}^d_s)$, were characterized completely by McMullen's "g-conjecture" [16], as proven by Billera & Lee [5] and Stanley [22].

The flag vector of a polytope, introduced by Bayer & Billera [4] as the "extended f-vector," in general contains considerably more combinatorial information than the f-vector. For example, the dimension of the affine hull of the set of flag vectors, which we denote by $f\ell(\mathcal{P}^d)$, is a Fibonacci number minus one [4], and thus grows exponentially with d. On the other hand, within some important classes of polytopes the f-vector determines the flag

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vector; for example, this holds for 3-polytopes, as well as for simplicial polytopes (and thus for simple polytopes, by duality). Thus the results just quoted also characterize the flag vector sets $f\ell(\mathcal{P}^3)$ and $f\ell(\mathcal{P}_s^d)$.

While the f-vectors and the flag vectors of 3-polytopes are thus completely understood, for 4-dimensional polytopes our knowledge about the sets of f-vectors and of flag vectors is rather incomplete. The 2-dimensional coordinate projections of the 3-dimensional set $f(\mathcal{P}^4)$ have been determined completely, see [13, Sect. 10.4 and p. 198c]. A systematic study of the 4-dimensional set $f\ell(\mathcal{P}^4)$ was started by Bayer [3]; see also [14] and [27] [28]. But there are still open questions, e.g. those related to the *fatness*. While there are 3-spheres of arbitrarily large fatness, this is not known for 4-polytopes [28]. On the other hand, there is a lower bound for the fatness of 4-polytopes, but we do not know whether this also holds for 3-spheres [28].

As polyhedral spheres appear very naturally in any attempt to enumerate combinatorial types of polytopes (see e.g. Grünbaum [13, Sects. 3.3, 5.5]), one must ask whether the characterizations of sets of f- resp. flag vectors extend to polyhedral spheres (that is, regular CW spheres such that any two faces intersect in a single face, which may be empty), that is, whether

$$f(\mathcal{S}^{d-1}) \;=\; f(\mathcal{P}^d) \qquad \text{and} \qquad f\ell(\mathcal{S}^{d-1}) \;=\; f\ell(\mathcal{P}^d)\ ? \tag{$*$}$$

Steinitz's theorem [24] [25] in essence proves that $\mathcal{S}^2 = \mathcal{P}^3$. The question whether $f(\mathcal{S}_s^{d-1}) = f(\mathcal{P}_s^d)$, or equivalently $f\ell(\mathcal{S}_s^{d-1}) = f\ell(\mathcal{P}_s^d)$, amounts to the "$g$-conjecture for spheres": McMullen's conjectured answer from [16] is "yes"; this is known to hold for $d \le 5$, as a consequence of the lower bound theorem for spheres, as proven by Barnette [2].

Up to now all available evidence with respect to the question ($*$) was positive. There are many simplicial 3-spheres (but also non-simplicial ones) that are non-polytopal, that is, not combinatorially equivalent to the boundary complex of a 4-polytope, so $\mathcal{P}_s^4 \subsetneqq \mathcal{S}_s^3$, and consequently $\mathcal{P}_s^d \subsetneqq \mathcal{S}_s^{d-1}$ for all $d \ge 4$. Indeed, *most* $(d-1)$-spheres are non-polytopal for $d \ge 4$, by comparison of Goodman & Pollack's upper bounds on the numbers of combinatorial types of polytopes [12] with the lower bounds for spheres by Kalai [15] and Pfeifle & Ziegler [21]. However, all the non-polytopal 3-spheres studied so far turned out to have an f-vector (and even flag vector) that is also the f- (resp. flag) vector of some 4-polytope: This was observed repeatedly, from the first examples (such as the Brückner and Barnette spheres, see e.g. Grünbaum [13, Sect. 11.5] and Ewald [11, Sect. III.4]) to the systematic enumerations of spheres with few vertices by Altshuler et al. (see e.g. [1] as well as [13, p. 96b]).

Here we establish, for the first time, that $f\ell(\mathcal{S}^{d-1}) = f\ell(\mathcal{P}^d)$ does not hold in general: Indeed, this fails for $d = 4$. For this we exhibit a specific flag vector in $f\ell(\mathcal{S}^3) \setminus f\ell(\mathcal{P}^4)$.

This flag vector belongs to a 2-simple 2-simplicial 3-sphere (*2s2s sphere*, for short): A 4-polytope or 3-sphere is 2*-simple* if every edge lies in exactly three facets, and 2*-simplicial* if every 2-face has exactly three vertices. The 2s2s property is closed under duality. The 2s2s 4-polytopes were introduced by Grünbaum [13, Sect. 4.5]. This fascinating class of polytopes includes the hypersimplex and the 24-cell. It appears naturally in the study of $f\ell(\mathcal{P}^4)$ (see Bayer [3], Ziegler [27] [28], and Paffenholz & Werner [19]). The flag vectors of 2s2s 3-spheres/ 4-polytopes are of the form

$$(f_0, f_1, f_2, f_3; f_{02}) \;=\; (n, m, m, n; 3m),$$

and any 3-sphere/4-polytope with such a flag vector is 2s2s. In particular, within the class 2s2s, the f-vector determines the flag vector. (Here and in the following we only list part of the full flag vector: All other components are determined by the generalized Dehn–Sommerville equations of Bayer & Billera [4].) However, there are 2s2s and non-2s2s polytopes which have the same f-vector: See Paffenholz & Ziegler [20, Cor. 4.3] for examples. Our main result is that there is a specific 2s2s 3-sphere that is non-polytopal and at the same time unique for its flag vector.

Theorem 1.1. *There is a unique polyhedral* 3*-sphere, but no convex* 4*-polytope, with flag vector given by*

$$(f_0, f_1, f_2, f_3; f_{02}) = (12, 40, 40, 12; 120).$$

Thus, the set of flag vectors of 4*-polytopes is a proper subset of the set of flag vectors of* 3*-spheres:*

$$f\ell(\mathcal{P}^4) \subsetneq f\ell(\mathcal{S}^3).$$

Proof. Any 3-sphere with the given flag vector is necessarily 2-simplicial, as $f_{02} = 3f_2$, and it is 2-simple as $f_{13} = f_{02} = 3f_1$. A 3-sphere W_{12}^{40} with this flag vector was discovered in 2009 by Werner [26, Table 7.1 left] in the course of his partial enumeration of shellable 2s2s 3-spheres with 12 vertices. In Section 2 we report about a complete enumeration of 2s2s 3-spheres with at most 12 vertices, which yields that Werner's 3-sphere W_{12}^{40} is the only sphere with this flag vector. Finally, in Section 3 we prove that this sphere does not have an oriented matroid, so in particular it is not polytopal. □

We would assume that also $f(\mathcal{P}^4) \subsetneq f(\mathcal{S}^3)$, but we have not proved that. For example, one might speculate that there is even no 4-polytope with the f-vector $(f_0, f_1, f_2, f_3) = (12, 40, 40, 12)$. As Marge Bayer has pointed out to us, any 3-sphere with this f-vector would be "close" to being 2s2s, as it must satisfy $120 = 3f_2 \leq f_{02} \leq 130$ by [3, Thm. 2 (3)].

We would also assume that $f\ell(\mathcal{P}^d) \subsetneq f\ell(\mathcal{S}^{d-1})$ holds for all $d > 4$, but again this does not seem to follow immediately from our results.

2 2s2s 3-spheres with few vertices

The goal of this section is to report about the proof of the following result, which includes the uniqueness claim in Theorem 1.1.

Theorem 2.1. *The following is a complete list of combinatorial types of* 2*-simple* 2*-simplicial* 3*-spheres with at most* 12 *vertices.*

#vert.	name	flag vector	reference	realization/polytope
5	Δ_5	$(5, 10, 10, 5; 30)$		simplex
9	W_9	$(9, 26, 26, 9; 78)$	[26]	[26, Thm. 4.2.2]
10	W_{10}	$(10, 30, 30, 10; 90)$	[19, Sect. 4.1]	[19, Sect. 4.1]
	$\Delta_4(2)$	$(10, 30, 30, 10; 90)$	[13, p. 65]	hypersimplex
	$\Delta_4(2)^*$	$(10, 30, 30, 10; 90)$		dual of $\Delta_4(2)$
11	P_{11}	$(11, 34, 34, 11; 102)$	[19, Sect. 4.1]	[19, Sect. 4.1]
12	W_{12}^{39}	$(12, 39, 39, 12; 117)$	[26, Tbl. 7.1 right]	[17, Sect. 4.2]
	W_{12}^{40}	$(12, 40, 40, 12; 120)$	[26, Tbl. 7.1 left]	none: see Sect. 3

All of these, except for the hypersimplex and its dual, are self-dual.

This improves upon results of Werner [26], who classified the 2s2s 3-polytopes with $f_0 \leq 9$ vertices [26, Thm. 7.2.13]. He also performed a computer enumeration that produced all *shellable* 2s2s 3-spheres with $f_0 \leq 11$. For $f_0 = 12$ Werner's computations remained incomplete due to constraints in computing power; however, his incomplete enumeration produced the two spheres mentioned above. One of these spheres, W_{12}^{39}, was realized as a 4-polytope by Miyata [17].

2.1 The enumeration algorithm

For a 3-sphere S we define the *p-vector* $p(S) = (p_4, p_5, \ldots)$, where p_i is the number of facets of S with i vertices. For any 2s2s 3-sphere with f-vector (n, m, m, n) we have $p_i = 0$ for $2i - 4 \geq n$, since a facet with i vertices is a simplicial 3-polytope with $2i - 4$ faces and thus has $2i - 4$ neighboring facets. In particular, for $n = 12$ we have $p_i = 0$ for $i > 7$. Moreover, we have $\sum_{i \geq 4} p_i = n$, and $\sum_{i \geq 4} (2i - 4) p_i = 2m$. This yields a finite list of possible p-vectors for any possible f-vector. For example, for $f = (12, 40, 40, 12)$ there are exactly 23 potential p-vectors that satisfy the three restrictions. To enumerate all 2s2s 3-spheres with a given number n of vertices, note that

$$2n \leq m \leq \tfrac{1}{4} n(n+3). \tag{M1}$$

While the lower bound is trivial, the upper bound stems from [3, Thm. 2 (3)].

We have designed and implemented an enumeration algorithm in order to produce, for each p-vector, one symmetry representative of each set system (of vertex sets of facets) that has the given p-vector and is *proper* in the sense that it satisfies

(I1) the intersection of two facets contains either 0, 1 or 3 vertices,
(I2) the intersection of three facets contains at most 2 vertices, and
(I3) the intersection of four facets contains at most 1 vertex.

This is where we crucially use the fact that we are looking for 2s2s 3-spheres only. The resulting lists are then checked for being Eulerian lattices of rank 5.

The idea for symmetry breaking in the enumeration, and thus for avoiding to produce re-labelled versions of the same facet lists too often, was to fix the labelling of the vertex set of a facet of maximal size i, and then to assign step by step vertex labels to a remaining facet of maximal size. It turned out that in some cases even more facets could be fixed, or at least had up to re-labelling only few distinct possibilities. In particular this was the case whenever $p_7 > 0$.

Algorithm 2.2. find_facet_lists(p)
INPUT: p-vector $(p_4, p_5, \ldots)$
OUTPUT: the facet lists of all 2s2s rank 5 Eulerian lattices with this p-vector up to combinatorial equivalence

(1) ind $= \max\{i : p_i > 0\}$
(2) facet_list $= \{\{0, \ldots, \text{ind} - 1\}\}$
(3) $p_{ind} = p_{ind} - 1$
(4) ind $= \max\{i : p_i > 0\}$
(5) stc $= \{\{i_0, \ldots, i_{ind}\}$: intersection with facet_list is proper$\}$

(6) for $F \in$ stc:
(7) facet_list = facet_list $\cup \{F\}$
(8) recursively add new facets to the list
(9) evaluate whenever there are enough facets in the list
(10) facet_list = facet_list $\setminus \{F\}$

After roughly two weeks of computation on standard linux workstations with altogether 45 kernels, the algorithm had enumerated all 2s2s rank 5 Eulerian lattices with up to 12 vertices. This produced exactly the face lattices of the spheres listed in Theorem 2.1, and thus proves that theorem as well as the second part of Theorem 1.1.

We refer to the works by Paffenholz and Werner [18] [19] [26] for information and data on 2s2s 4-polytopes with more than 12 vertices.

3 Non-polytopality

Our approach to prove non-polytopality is via oriented matroids. This is a standard method for proving the non-realizability of polytopes as well as of polyhedral surfaces (see for example Bokowski & Sturmfels [9], Bokowski [7], Björner et al. [6, Chap. 8]), but as far as we know this has always been applied to simplicial polytopes or surfaces, and thus in a setting of uniform oriented matroids, with the notable exception of Bremner's software package `mpc` [10], see Bokowski, Bremner & Gévay [8, Sect. 7]. (Indeed, David Bremner has confirmed the non-existence result of this section using the `nuoms` function of his package.) Here we demonstrate that the oriented matroid method is particularly effective in an example with a non-simplicial sphere, and hence for a non-uniform oriented matroid.

The basic approach is as follows: Any set of points $v_0, \dots, v_N \in \mathbb{R}^d$ leads to an orientation function $\chi : \{v_0, v_1, \dots, v_N\}^{d+1} \to \{0, +1, -1\}$ by setting

$$\chi(v_{i_0}, v_{i_1}, \dots, v_{i_d}) \;:=\; \operatorname{sign}\det \begin{pmatrix} v_{i_0} & v_{i_1} & \cdots & v_{i_d} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

This map is a *chirotope of rank* $d+1$. In addition to the condition that its support has to be a matroid (which we do not use; cf. [6, Thm. 3.6.2]), this means that

(C1) it is alternating, and
(C2) it satisfies the *three term Grassmann–Plücker relations*: For any $d-1$ points $\lambda = (v_{i_0}, \dots, v_{i_{d-2}})$ and four points v_a, v_b, v_c, v_d the set

$$\{\, \chi(\lambda, v_a, v_b) \cdot \chi(\lambda, v_c, v_d),\; -\chi(\lambda, v_a, v_c) \cdot \chi(\lambda, v_b, v_d),\; \chi(\lambda, v_a, v_d) \cdot \chi(\lambda(v_b, v_c) \,\}$$

either equals $\{0\}$ or contains $\{-1, +1\}$.

If the points $v_0, \dots, v_N$ are supposed to be the vertices of a d-dimensional polytope with a prescribed facet list $(F_1, \dots, F_n)$, then the map must satisfy the following extra conditions:

(P1) If $v_{i_0}, \dots, v_{i_d}$ are contained in a facet F_j, then $\chi(v_{i_0}, \dots, v_{i_d}) = 0$.
(P2) If $v_{i_1}, \dots, v_{i_d}$ are contained in a facet F_j which does not contain v_a or v_b, then

$$\chi(v_a, v_{i_1}, \dots, v_{i_d}) = \chi(v_b, v_{i_1}, \dots, v_{i_d}).$$

Proof of Theorem 1.1 (the non-polytopality part). Werner's sphere W_{12}^{40} with the flag vector $(f_0, f_1, f_2, f_3; f_{02}) = (12, 40, 40, 12; 120)$ is given by the following list of facets (where we use the vertex labeling of [26, Table 7.1, left]):

$F_1 : \{v_0, v_1, v_2, v_3\}$	$F_7 : \{v_0, v_5, v_7, v_8, v_9, v_{10}\}$
$F_2 : \{v_0, v_2, v_3, v_4, v_5, v_6, v_7\}$	$F_8 : \{v_1, v_2, v_3, v_4, v_{10}, v_{11}\}$
$F_3 : \{v_0, v_1, v_3, v_4, v_8, v_9\}$	$F_9 : \{v_2, v_5, v_6, v_8, v_{10}, v_{11}\}$
$F_4 : \{v_0, v_1, v_2, v_6, v_9, v_{10}\}$	$F_{10} : \{v_1, v_8, v_9, v_{10}, v_{11}\}$
$F_5 : \{v_0, v_4, v_7, v_8\}$	$F_{11} : \{v_1, v_4, v_5, v_7, v_8, v_{11}\}$
$F_6 : \{v_0, v_5, v_6, v_{10}\}$	$F_{12} : \{v_2, v_4, v_5, v_{11}\}$

In order to prove non-polytopality of this sphere, we will show that there is no chirotope compatible with its facet list.

In the sphere, $\{v_8, v_9, v_{10}\} = F_7 \cap F_{10}$ is the vertex set of a triangle 2-face, so the vertices $\{v_7, v_8, v_9, v_{10}\} \subset F_7$ span a tetrahedron, while $v_2 \notin F_7$. Thus in any realization, we have $\chi(v_7, v_8, v_{10}, v_2, v_9) \neq 0$. Thus we may fix an orientation of the realization by setting $\chi(v_7, v_8, v_{10}, v_2, v_9) := +1$. Starting with this, we obtain the implications

	$\chi(v_7, v_8, v_{10}, v_2, v_9) = +1$	
$F_7 = \{v_0, v_5, v_7, v_8, v_9, v_{10}\}$	$\chi(v_7, v_8, v_{10}, v_{11}, v_9) = +1$	
	$\chi(v_7, v_8, v_{10}, v_4, v_9) = +1$	$\{v_1, v_8, v_9, v_{10}, v_{11}\} = F_{10}$
	$\chi(v_7, v_8, v_{10}, v_1, v_9) = +1$	
	$\chi(v_8, v_{10}, v_{11}, v_2, v_9) = -1$	
	$\chi(v_7, v_8, v_{10}, v_{11}, v_2) = -1$	$F_9 = \{v_2, v_5, v_6, v_8, v_{10}, v_{11}\} = F_9$
$F_{10} = \{v_1, v_8, v_9, v_{10}, v_{11}\}$	$\chi(v_4, v_8, v_{10}, v_{11}, v_2) = -1$	
	$\chi(v_8, v_{10}, v_2, v_1, v_9) = +1$	
	$\chi(v_8, v_{10}, v_0, v_1, v_9) = +1$	$\{v_0, v_1, v_2, v_6, v_9, v_{10}\} = F_4$
$F_3 = \{v_0, v_1, v_3, v_4, v_8, v_9\}$	$\chi(v_4, v_{10}, v_2, v_1, v_9) = +1$	$\{v_1, v_2, v_3, v_4, v_{10}, v_{11}\} = F_8$
	$\chi(v_4, v_8, v_{10}, v_2, v_1) = -1$	
$F_7 = \{v_0, v_5, v_7, v_8, v_9, v_{10}\}$	$\chi(v_8, v_7, v_0, v_1, v_9) = +1$	
	$\chi(v_8, v_7, v_0, v_4, v_9) = +1$	
$F_5 = \{v_0, v_4, v_7, v_8\}$	$\chi(v_8, v_7, v_0, v_4, v_5) = +1$	
	$\chi(v_8, v_7, v_0, v_4, v_{11}) = +1$	
	$\chi(v_8, v_7, v_0, v_4, v_1) = +1$	$\{v_1, v_4, v_5, v_7, v_8, v_{11}\} = F_{11}$
$F_{11} = \{v_1, v_4, v_5, v_7, v_8, v_{11}\}$	$\chi(v_4, v_8, v_{10}, v_7, v_{11}) = +1$	$\{v_0, v_2, v_3, v_4, v_5, v_6, v_7\} = F_2$
	$\chi(v_4, v_8, v_{10}, v_7, v_1) = +1$	
	$\chi(v_{11}, v_7, v_0, v_4, v_5) = +1$	$\{v_1, v_4, v_5, v_7, v_8, v_{11}\} = F_{11}$
	$\chi(v_{11}, v_7, v_2, v_4, v_5) = +1$	$\{v_2, v_4, v_5, v_{11}\} = F_{12}$
	$\chi(v_{11}, v_1, v_2, v_4, v_5) = +1$	$\{v_1, v_2, v_3, v_4, v_{10}, v_{11}\} = F_8$
	$\chi(v_{11}, v_1, v_2, v_4, v_8) = +1$	$\{v_1, v_4, v_5, v_7, v_8, v_{11}\} = F_{11}$
	$\chi(v_4, v_8, v_{10}, v_{11}, v_1) = +1$	

Note that this chain of arguments uses all facets except for F_1 and F_6; all vertices occur except for v_3 and v_6.

Given the values of χ that we have obtained, the contradiction appears in the three term Grassmann–Plücker relations:
Let $\lambda_1 = (v_7, v_8, v_{10})$, $a_1 = v_4$, $b_1 = v_{11}$, $c_1 = v_2$, $d_1 = v_9$,
and $\lambda_2 = (v_4, v_8, v_{10})$, $a_2 = v_7$, $b_2 = v_{11}$, $c_2 = v_2$, $d_2 = v_1$.
Then using the marked values of χ we get

$$\begin{aligned}\{\,\chi(\lambda_1,a_1,b_1)\cdot\chi(\lambda_1,c_1,d_1),\quad -\chi(\lambda_1,a_1,c_1)\cdot\chi(\lambda_1,b_1,d_1),\quad \chi(\lambda_1,a_1,d_1)\cdot\chi(\lambda_1,b_1,c_1)\,\} &=\\ \{(-1)\cdot(+1),\quad -\chi(v_7,v_8,v_{10},v_4,v_2)\cdot(+1),\quad (+1)\cdot(-1)\};&\\ \{\,\chi(\lambda_2,a_2,b_2)\cdot\chi(\lambda_2,c_2,d_2),\quad -\chi(\lambda_2,a_2,c_2)\cdot\chi(\lambda_2,b_2,d_2),\quad \chi(\lambda_2,a_2,d_2)\cdot\chi(\lambda_2,b_2,c_2)\,\} &=\\ \{(+1)\cdot(-1),\quad -\chi(v_4,v_8,v_{10},v_7,v_2)\cdot(+1),\quad (+1)\cdot(-1)\}.&\end{aligned}$$

Thus both sets contain -1, while by the alternating property of χ not both of them can contain $+1$. Therefore, there is no chirotope and hence no oriented matroid for the sphere W_{12}^{40}, so it is not polytopal. □

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