Tverberg plus constraints

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Tverberg's celebrated theorem from 1966 [7], its topological version by Bárány, Shlosman & Szűcs [1] and Özaydin [4] 1981/1987, and the 2009 optimal colored version by Blagojević, Matschke & Ziegler [3], may together be summarized as follows:

Theorem (Tverberg [7]; topological Tverberg [1] [4]; optimal colored Tverberg [3])**.** *Let $d \geq 1$ and $r \geq 2$, and let $N \geq N_0 := (r-1)(d+1)-1$.*

(1) *For any affine map $f : \Delta_N \to \mathbb{R}^d$ the N-dimensional simplex Δ_N contains r points $x_1, \dots, x_r$ that lie in r vertex-disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ whose images coincide: $f(x_1) = \cdots = f(x_r)$.*

(2) *If $r \geq 2$ is a prime power, then this holds more generally for arbitrary continuous maps.*

(3) *If $r \geq 2$ is a prime, then the faces σ_i may in addition be required to have distinct vertex colors (that is, to be "rainbow faces") for any coloring of the vertex set of Δ_N whose color classes have size at most $r-1$.*

In our lecture we presented a new, simple and elementary, proof technique (see [2]) that establishes many of the known extensions of these theorems for maps of a simplex Δ_N *of higher dimension* N directly from these *optimal* results for $N = N_0$. Indeed, we even obtain sharpened and improved results from this.

Our new technique relies on a concept of "Tverberg unavoidable subcomplexes":

Definition 1 (Tverberg unavoidable subcomplexes)**.** Let $r \geq 2$, $d \geq 1$, and N be integers, and let $f : \Delta_N \to \mathbb{R}^d$ be a continuous map with at least one Tverberg r-partition. Then a subcomplex $\Sigma \subseteq \Delta_N$ is *Tverberg unavoidable* if for every Tverberg partition $\{\sigma_1, \dots, \sigma_r\}$ for f there is at least one face σ_j that lies in Σ.

Examples of Tverberg unavoidable complexes are obtained using the pigeonhole principle. For example, for any set S of at most $2r-1$ vertices in Δ_N the subcomplex of faces with at most one vertex in S is Tverberg unavoidable. And if $r(k+2) > N+1$, then the k-skeleton $\Delta_N^{(k)}$ of Δ_N is Tverberg unavoidable.

If N is large enough, then we can require points of Tverberg coincidence to equalize additional constraint functions, via maps to an extended target space:

Lemma 2 (Key lemma #1)**.** *Let r be a prime power, $d \geq 1$, and $c \geq 0$. Let $N \geq N_c := (r-1)(d+1+c)$ and let $f : \Delta_N \to \mathbb{R}^d$ and $g : \Delta_N \to \mathbb{R}^c$ be continuous. Then there are r points $x_i \in \sigma_i$, where $\sigma_1, \dots, \sigma_r$ are pairwise disjoint faces of Δ_N with $g(x_1) = \cdots = g(x_r)$ and $f(x_1) = \cdots = f(x_r)$.*

Using the symmetry of the problem, we now obtain a Tverberg partition that does not only have one, but *all* of its faces in the unavoidable subcomplex:

Lemma 3 (Key lemma #2)**.** *Let r be a prime power, $d \geq 1$, and $N \geq N_1 = (r-1)(d+2)$. Assume that $f : \Delta_N \to \mathbb{R}^d$ is continuous and that the subcomplex*

$\Sigma \subseteq \Delta_N$ is Tverberg unavoidable for f. Then there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N, all of them contained in Σ, such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.

For example, our Ansatz produces directly from the topological Tverberg theorem a colored version that is stronger than Živaljević & Vrećica's 1992 colored Tverberg's theorem [9] that is valid for all prime powers r:

Theorem 4 (Weak colored Tverberg)**.** *Let r be a prime power, $d \geq 1$, $N \geq N_{d+1} = (r-1)(2d+2)$ and let $f : \Delta_N \to \mathbb{R}^d$ be continuous. If the vertices of Δ_N are colored by $d+1$ colors, where each color class has cardinality at most $2r-1$, then there are r pairwise disjoint rainbow faces $\sigma_1, \dots, \sigma_r$ of Δ_N such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

Similarly we obtain directly from the topological Tverberg theorem a strengthened version of the generalized van Kampen–Flores theorem of Sarkaria [5] and Volovikov [8] from 1991/1996:

Theorem 5 (Generalized van Kampen–Flores)**.** *Let r be a prime power, $d \geq 1$, $N \geq N_1 = (r-1)(d+2)$, and $k \geq \lceil \frac{r-1}{r} d \rceil$. Then for every continuous $f : \Delta_N \to \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N, with $\dim \sigma_i \leq k$ for $1 \leq i \leq k$, such that $f(\sigma_1) \cap \dots \cap f(\sigma_r) \neq \emptyset$.*

As another example, our machinery reproves Soberón's 2013 Tverberg theorem with equal barycentric coordinates and at the same time produces a new topological version of this result.

Details appear in our paper [2].

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