Simplicial complex models for arrangement complements

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1. Why do we care?

1.1. **Arrangements and configuration spaces.** The configuration space of n labeled distinct points on a manifold

$$F(X,n) := \{(x_1,\dots,x_n) \in X^n : x_i \neq x_j \text{ for } i<j\}$$

appears in diverse contexts in topology (providing, for example, embedding invariants and models for loop spaces), knot theory, and physics (KZ equation, renormalization); see e.g. Vassiliev [20] and Fadell & Husseini [8]. In particular, the space

$$F(\mathbb{R}^d,n) = \{(x_1,\dots,x_n) \in \mathbb{R}^{d\times n} : x_i \neq x_j \text{ for } i<j\}$$

has been studied in great detail. It is the complement of an arrangement of linear codimension d subspaces in $\mathbb{R}^d$ whose intersection lattice (with the customary ordering by reversed inclusion) is the partition lattice Π_n of rank $n-1$. The cohomology is free, with Poincaré polynomial $\prod_{i=1}^{n-1}(1+i\,t^{d-1})$; see e.g. Björner [4], Goresky & MacPherson [13, Part III].

In particular, for $d=2$ the space $F(\mathbb{C},n) = F(\mathbb{R}^2,n)$ appears in work by Arnol'd related to Hilbert's 13th problem, continued in papers by works by Fuks, Deligne, Orlik & Solomon, and many others. It is a key example for the theory of (complex) hyperplane arrangements; see e.g. Orlik & Terao [17].

The space $F(\mathbb{R}^d,n)$ is the complement of a codimension d subset in $\mathbb{R}^{d\times n}$, so in particular it provides a $(d-2)$-connected space on which the symmetric group $\mathfrak{S}_n$ acts freely. Thus the inclusions $F(\mathbb{R}^2,n) \subset F(\mathbb{R}^3,n) \subset \cdots$ can be used to compute the cohomology of $\mathfrak{S}_n$; see Giusti & Sinha [12] for recent work, which is based on the Fox–Neuwirth stratification [11] of the configuration spaces $F(\mathbb{R}^d,n)$.

1.2. **Cell complex models.** A problem by R. Nandakumar and N. Ramana Rao [16] asks whether every bounded convex set P in the plane can be divided into n convex pieces that have equal area and equal perimeter. In [18] the same authors prove this for $n=2^k$ in the case where P is a convex polygon. Blagojević, Bárány & Szűcs [2] established the problem for $n=3$.

Karasev [15] and Hubard & Aronov [14] observed that a positive solution for the problem would — via optimal transport (cf. Villani [21]) and generalized Voronoi diagrams (cf. Aurenhammer et al. [1]) — follow from the non-existence of an equivariant map

$$F(\mathbb{R}^2,n) \longrightarrow_{\mathfrak{S}_n} S(\{(y_1,\dots,y_n)\in\mathbb{R}^n : y_1+\cdots+y_n=0\}) \simeq S^{n-2}.$$

A d-dimensional and more general version of the problem, to partition any sufficiently continuous measure on $\mathbb{R}^d$ into n pieces of equal measure that also equalize $d-1$ further continuous functions, could be solved by the non-existence of

$$F(\mathbb{R}^d,n) \longrightarrow_{\mathfrak{S}_n} S(\{(y_1,\dots,y_n)\in\mathbb{R}^{(d-1)\times n} : y_1+\cdots+y_n=0\}) \simeq S^{(n-1)(d-1)-1}.$$

In the cited works by Karasev and Hubard & Aronov this is approached via (twisted) Euler class computations on the one-point compactification of $F(\mathbb{R}^d, n)$ (which is not a manifold), which leads to the non-existence of these maps if n is a prime power.

Here we report about an alternative approach, via Equivariant Obstruction Theory (as developed by tom Dieck [7, Sect. II.3]). For this we need an equivariant cell complex model for $F(\mathbb{R}^d, n)$.

2. A method

We rely on a method developed in Björner & Ziegler [5] to obtain a compact cell complex model for the complements of linear subspace arrangements. For this let $\mathcal{A}$ be a finite arrangement of linear subspaces in a real vector space $\mathbb{R}^N$. Each k-dimensional subspace F of the arrangement is embedded into a complete flag of linear subspaces $F = F_k \subset F_{k+1} \subset \cdots \subset \mathbb{R}^N$. The union of these flags yields a stratification of $\mathbb{R}^N$ into relative-open convex cones. These cones are not usually pointed, but their faces are unions of strata. The barycentric subdivision of the stratification yields a triangulation of a star-shaped neighborhood of the origin in $\mathbb{R}^N$, which as a subcomplex contains a triangulation of the link of the arrangement. Its Alexander dual is the barycentric subdivision of a regular CW complex, realized as a geometric simplicial complex that is a *strong deformation retract of the complement.* Moreover, if the arrangement has a symmetry, and the flags are chosen to be compatible with the symmetry, then the resulting complex carries the symmetry of the arrangement.

3. Examples

Implementing the construction from [5] yields the Fox–Neuwirth stratification on the complement $F(\mathbb{R}^d, n)$ of the arrangement, but indeed our construction and proof uses the stratification on the full ambient space $\mathbb{R}^{d\times n}$.

Theorem 3.1. *There is a regular cell complex $\mathcal{F}(d,n)$ of dimension $(n-1)(d-1)$ that has $n!$ vertices and $n!$ facets (maximal cells), with a free cellular action of the symmetric group $\mathfrak{S}_n$ that is transitive on the vertices as well as on the facets.*

The barycentric subdivision $\mathrm{sd}\mathcal{F}(d,n)$ *has a geometric realization in $F(\mathbb{R}^d, n)$ as an equivariant strong deformation retract.*

Based on this model, our Equivariant Obstruction Theory calculation gives a complete answer to the equivariant map problem, and thus a simple combinatorial proof for the prime power case of the Nandakumar & Ramana Rao conjecture:

Theorem 3.2 ([6])**.** *An equivariant continuous map*

$$F(\mathbb{R}^d, n) \longrightarrow_{\mathfrak{S}_n} S(\{(y_1,\dots,y_n) \in \mathbb{R}^{(d-1)\times n} : y_1+\cdots+y_n = 0\}) \simeq S^{(n-1)(d-1)-1}.$$

does not exist if and only if n is a prime power.

At the combinatorial core of our calculation lies the fact, apparently first proved by B. Ram in 1909 [19], that $\gcd\{\binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n-1}\}$ equals p for any prime power $n = p^k$, and equals 1 otherwise.

4. Further Examples

In view of further applications to geometric measure partition problems, there is interest in constructing and analyzing cell complex models for spaces such as $F(S^d, n)$ (see Feichtner & Ziegler [9] and Basabe et al. [3]) as well as $F_{\pm}(S^d, n)$ (see [10]).

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