Polytopes with low-dimensional realization spaces

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(joint work with Karim Adiprasito)

We discuss, dimension-by-dimension, the two questions

Question 1: *Is it true that, up to projective transformations, there are only finitely many combinatorial types of d-dimensional convex polytopes?*

(This was conjectured to be true by Shephard and McMullen in the sixties; compare [4].)

Question 2: *How does, for d-dimensional polytopes, the dimension of the realization space grow with the size of the polytopes?*

(Suitable definitions of "realization space" and "size" are given below.)

1. Definitions

Let P be a convex d-polytope of *size* $\text{size}(P) := f_0(P) + f_{d-1}(P)$. We define its *(centered) realization space* as

$$\mathcal{R}_0(P) := \big\{(V,C) \in \mathbb{R}^{d\times(f_0+f_{d-1})} : \text{conv}(V) = \{x \in \mathbb{R}^d : C^t x \leq \mathbf{1}\}\text{realizes}P\big\},$$

that is, as the set of combined vertex and facet descriptions of realizations of P that have the origin in the interior. Note that $(C,V) \in \mathbb{R}^{d\times(f_0+f_{d-1})}$ lies in this set iff $c_i^t v_j = 1$ whenever v_j lies on the facet $F_i = \{x \in \mathbb{R}^d : c_i^t x = 1\}$ and $c_i^t v_j < 1$ otherwise. Thus this realization space is a primary semialgebraic set in $\mathbb{R}^{d\,\text{size}(P)}$; it particular, its *dimension* is well-defined.

The polytope P is called *projectively unique* if the group $\text{PGL}(\mathbb{R}^d)$ of projective transformations on $\mathbb{R}^d$ acts transitively on $\mathcal{R}_0(P)$. In particular, for projectively unique polytopes we have $\dim \mathcal{R}_0(P) \leq \dim \text{PGL}(\mathbb{R}^d) = d(d+2)$, with equality in all non-trivial cases (if the vertex set of the polytope contains a projective basis, that is, it is not a join).

The *naive guess* for the dimension of the (centered) realization space is

$$\text{NG}(P) := d(f_0 + f_{d-1}) - f_{0,d-1},$$

where $f_{0,d-1}$ is the number of vertex-facet incidences. This quantity is "the number of variables minus the number of equations" in the above system. (Compare [1].)

2. Examples

For $d = 2$, the only projectively unique 2-polytopes are triangle Δ_2 and square $\square$.

For $d = 3$, the projectively unique 3-polytopes are the four types of 3-polytopes with at most 9 edges, that is: tetrahedron Δ_3, square pyramid $\square * \Delta_0$, prism over triangle $\Delta_2 \times \Delta_1$, bipyramid over triangle $\Delta_2 \oplus \Delta_1$ according to Grünbaum [2, Exercise 4.8.30]. Indeed, this is compatible with the result of Steinitz [9] (see also Richter-Gebert [6, Sect. 13.3]) that the dimension of the realization space $\mathcal{R}_0(P)$ of a 3-polytope is $\text{NG}(P) = f_1 + 6 = f_1 - 9 + \dim \text{PGL}(\mathbb{R}^d)$.

This answers Questions 1 and 2 for $d \leq 3$: For Question 1 the answer is yes, and *the dimension of the realization space grows linearly with the size.*

3. Universality

The naive guess $\mathrm{NG}(P)$ yields the dimension of the (centered) realization space correctly for all polytopes of dimension $d \leq 3$, as well as for all simple or simplicial polytopes. However, Robertson's [7] claim that $\mathcal{R}_0(P)$ is always a differentiable manifold of dimension $\mathrm{NG}(P)$ is far from being true — indeed, according to Mnëv's universality theorem [5] [6], $\mathcal{R}_0(P)$ is "stably equivalent" to an arbitrary primary semialgebraic set defined over $\mathbb{Z}$; here "stable equivalence" means that we loose the control over the dimension, but keep singularities as well as the (non)existence of rational points. Thus, we cannot assume that $\mathrm{NG}(P)$ yields the dimension of the realization space in general — the guess could be way too high, or too low. Indeed, according to Richter-Gebert [6] this already happens for $d = 4$.

4. Shephard's list

Construction methods for projectively unique d-polytopes were developed by Peter McMullen in his doctoral thesis (Birmingham 1968) directed by G. C. Shephard; see [4], where McMullen writes: "Shephard (private communication) has independently made a list, believed to be complete, of the projectively unique 4-polytopes. All of these polytopes can be constructed by the methods described here."

If the conjecture is correct, then the following list of eleven projectively unique 4-polytopes (all of them generated by McMullen's techniques, duplicates removed) should be complete:

Construction		$(f_0, f_1, f_2, f_3; f_{03})$	NG	facets
Δ_4	selfdual	(5,10,10,5;20)	20	5 tetrahedra
$\Box * \Delta_1$	selfdual	(6,11,11,6;26)	22	4 tetrah., 2 square pyramids
$(\Delta_2 \oplus \Delta_1) * \Delta_0$		(6,14,15,7;29)	23	6 tetrah., 1 bipyramid
$(\Delta_2 \times \Delta_1) * \Delta_0$		(7,15,14,6;29)	23	2 tetrah., 3 sq. pyr., 1 prism
$\Delta_3 \oplus \Delta_1$	simplicial	(6,14,16,8;32)	24	8 tetrah.
$\Delta_3 \times \Delta_1$	simple	(8,16,14,6;32)	24	2 tetrah., 4 prisms
$\Delta_2 \oplus \Delta_2$	simplicial	(6,15,18,9;36)	24	9 tetrah.
$\Delta_2 \times \Delta_2$	simple	(9,18,15,6;36)	24	6 prisms
$(\Box, v) \oplus (\Box, v)$		(7,17,18,8;36)	24	4 square pyramids, 4 tetrah.
... its dual		(8,18,17,7;36)	24	2 prisms, 4 sq. pyr., 1 tetrah.
v.split$(\Delta_2 \times \Delta_1)$	selfdual	(7,17,17,7;32)	24	3 tetrah., 2 sq. pyr., 2 bipyr.

5. Neighborly cubical polytopes

The *neighborly-cubical* 4-*polytopes* $\mathrm{NCP}_4(n)$ constructed by Joswig & Ziegler [3] and further analyzed by Sanyal & Ziegler [8] are cubical 4-polytopes with the graph of the n-cube. From these data, one can derive that the extended f-vector is

$$(f_0, f_1, f_2, f_3; f_{03}) = \tfrac{1}{4}2^n(4, 2n, 3(n-2), n-2, 8n-16).$$

In particular, we get that the naive guess is naive: $\mathrm{NG}(\mathrm{NCP}_4(n)) = 2^n(6-n)$ is negative for high n.

Nevertheless the neighborly-cubical polytopes are not projectively unique, but the dimension of the realization space is very small compared to the size:

Theorem. *The dimension of the realization space of* $\mathrm{NCP}_4(n)$ *grows quadratically in* n*, and thus only logarithmically in* $\mathrm{size}(\mathrm{NCP}_4(n))$.

The quadratic lower bound follows from the construction as a generic projection of a simple n-polytope with $2n$ facets. The upper bound uses the "cubical Gale evenness criterion" combinatorial description of $\mathrm{NCP}_4(n)$ from [3] and [8], and then a suitable ordering of the *vertices and facets*, such that after a quadratic number variables has been fixed, all further vertices and facets are determined when they occur in the ordering. We note that this proof technique can establish that a realization space is low-dimensional only if the naive guess is low as well.

6. Complexity

Candidate 4-polytopes that *could* have a low-dimensional realization space (or even be projectively unique) should thus have a low (or even negative) naive guess. We note that this is closely related to the "complexity" parameter for polytopes

$$C(P) := \frac{f_{0,3} - 20}{f_0 + f_3 - 10}$$

introduced in [10] being large; specifically, it should be at least 4. The only examples we seem to know for this in the moment are the neighborly cubical polytopes, and more generally the "projected deformed products of polygons" of [11] [8]. However, all these are not projectively unique, by construction.

References

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