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Abstracts

Face numbers of centrally-symmetric polytopes: Conjectures, Examples, Counterexamples

Günter M. Ziegler
(joint work with Raman Sanyal, Axel Werner)

The f-vectors of centrally-symmetric convex polytopes are the subject of three conjectures A, B, C of increasing strength by Kalai [4] from 1989. Such basic open questions illustrate that our understanding of the f-vectors of centrally-symmetric polytopes is dramatically incomplete. (Our understanding of f-vectors of general convex polytopes is also quite limited outside the range of simple/simplicial polytopes; compare [7], [8].)

In our lecture, based on [6], we described the three conjectures, and reported that Conjectures A and B hold for $d \leq 4$, while Conjecture C fails for $d \geq 4$ and Conjecture B fails for $d \geq 5$.

1. The case $d = 3$

The case of $d = 3$ is easy, but it solves as a model for a complete answer: The set of f-vectors of centrally-symmetric 3-polytopes is

$$\begin{aligned} \mathcal{F}_3^{cs} \quad = \quad \{(f_0, f_1, f_3) \in (2\mathbb{Z})^3 : \quad & f_0 - f_1 + f_2 = 2, \\ & f_0 \leq 2f_2 - 4, \\ & f_2 \leq 2f_0 - 4, \\ & f_0 + f_2 \geq 14 \ \}. \end{aligned}$$

2. Three conjectures

Kalai [4] described the following three conjectures of increasing strength about the f-vectors of 3-dimensional centrally-symmetric polytopes.

The first one, **Conjecture A**, claims that every such polytope has at least 3^d non-empty faces,

$$\sum_{i=0}^{d} f_i \ \geq \ 3^d.$$

This became known as the 3^d-*conjecture.* In its strong form, it would claim that equality occurs only for *Hanner polytopes*, which arise from 1-polytopes (intervals $[-1, 1]$) by repeated application of "taking products" and dualization.

The second one, **Conjecture B**, claimed that the f-vector of every centrally symmetric d-polytope P should componentwise dominate the f-vector of one of the Hanner polytopes, $f(P) \geq f(H)$.

The third one, **Conjecture C**, claimed that the flag-vector *flag*(P) of every centrally symmetric d-polytope P is dominated in flag-vector space by some flag vector *flag*(H), not only componentwise, but with respect to all linear flag-vector functionals that are nonnegative on all flag-vectors of general d-polytopes.

Kalai noted that quite obviously Conjecture C implies Conjecture B, which in turn implies the "3^d-conjecture", Conjecture A.

3. The cases 4(A) and 4(B)

While all three conjectures clearly hold for $d \leq 3$, we report that Conjectures A and B also hold for $d = 4$. The proof involves simple f-vector combinatorics, known elementary inequalities, some case distinctions, and one crucial non-trivial inequality, $g_2^{tor} \geq 2$. In its more general form for d-polytopes,

$$g_2^{tor}(P) \;=\; f_1 + f_{02} - 3f_2 - df_0 - \binom{d+1}{2} \;\geq\; \binom{d}{2} - d,$$

this inequality was derived by a Campo [1] via toric geometry. Following a suggestion by Kalai, we also derive an elementary proof via rigidity theory in [6].

4. Examples

As noted by Kalai, the Hanner polytopes (introduced by Hanner [3] in 1956, described above) provide a first, very interesting class of examples. A second class was described by Hansen [3] in 1977: The antiprisms over the independence polytopes of self-dual perfect graphs yield self-dual centrally-symmetric polytopes with interesting f-vectors. None of the two classes includes the other one: For examples take the sum of two 3-cubes, resp. the Hansen polytope of the path on 4 vertices. Both classes are examples of *weak Hanner polytopes* as introduced by Hansen, which have the property that any pair of opposite facets includes all the vertices. The hypersimplex $\Delta(k, 2k)$ of dimension $k-1$ is an example of a weak Hanner polytope that is neither Hanner nor Hansen in general.

5. The case 4(C) fails

Consider the flag vector functional

$$\alpha(P) \;:= (f_{02} - 3f_2) + (f_{13} - 3f_1),$$

which is non-negative, and vanishes exactly if P is 2-simplicial (first term) and 2-simple (second term).

This functional takes the values 9 and 12 on the 4-dimensional Hanner polytopes. Examples of centrally-symmetric 2-simplicial 2-simple 4-polytopes include Schläfli's 24-cell. Infinite families, which may also be obtained to be centrally symmetric, are described in [5].

Thus for $d = 4$ Conjecture C fails strongly, in the sense that there are infinitely many polytopes whose flag-vectors are separated from *all* flag-vectors of Hanner polytopes by a common nonnegative linear functional.

6. The cases 5(B) and 5(C) fail

For $d = 5$, we consider the linear f-vector functional

$$\beta(P) \ := \ f_0 + f_4.$$

This functional satisfies $\beta \geq 36$ on all Hanner polytopes, while $\beta = 32$ both for the Hansen polytope associated with the path on 4 vertices, with f-vector $(16, 64, 98, 64, 16)$, and on the central hypersimplex $\Delta(6, 3)$, whose f-vector is $(20, 90, 120, 60, 12)$.

Thus for $d = 5$ Conjecture B fails strongly, in the sense that there are are polytopes whose f-vectors are separated from *all* f-vectors of Hanner polytopes by a common nonnegative linear functional.

This implies, that indeed Conjecture C fails for all $d \geq 4$, and Conjecture B fails for all $d \geq 5$. Conjecture A remains open for $d \geq 5$.

We refer to [6] for details.

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