Abstracts

Combinatorial and polyhedral surfaces

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(joint work with Raman Sanyal and Thilo Schröder)

1. What is a surface?

There are several different combinatorial and geometric notions of a "polyhedral surface." *Topologically*, we consider the connected, orientable 2-manifold without boundary of genus $g \geq 0$, denoted M_g.

In the *combinatorial* version, we look at regular cell decompositions of M_g, which may be obtained by drawing graphs on the surfaces M_g, or by combinatorial prescriptions that tell us how to glue the surface from polygons. The "rotation schemes" of Heffter [6], see also Ringel [11], fall into this category. In the following, we will insist on the *intersection condition* to hold, according to which the intersection of any two cells of the surface consists of either one single edge, or one vertex, or is empty.

In the *geometric* version, a polyhedral surface is a complex formed by flat convex polygons, represented without intersections in $\mathbb{R}^3$, or in some $\mathbb{R}^N$.

See [4] and [13] for more detailed discussions of these models.

2. f-Vectors

For any (combinatorial or geometric) surface the *f-vector* (f_0, f_1, f_2) records the number of vertices, edges, and 2-faces. It thus also measures the topological complexity of the surface, whose genus is given by

$$g \quad = \quad 1 + \tfrac{1}{2}(f_1 - f_0 - f_2),$$

and the combinatorial complexity, via the average vertex degree and the average face degree, given by

$$\delta = \tfrac{2f_1}{f_0} \qquad \text{and} \qquad \delta^* = \tfrac{2f_1}{f_2}.$$

A key problem asks to characterize the f-vectors of combinatorial resp. geometric surfaces, and thus distinguish the two models in terms of their combinatorial characteristics.

For $g = 0$, a little lemma of Steinitz [12] characterizes the f-vectors by

$$f_0 - f_1 + f_2 = 2, \qquad 2f_1 \geq 3f_0, \qquad 2f_1 \geq 3f_2$$

for *both* models. Indeed, these are the f-vectors of convex 3-polytopes.

In contrast, for $g > 0$ the inequality $2f_1 \geq 3f_0$ is tight for combinatorial surfaces, while it is strict for geometric surfaces: A geometric surface satisfying $2f_1 = 3f_0$ is necessarily realized in $\mathbb{R}^3$, and convex. Indeed, we have $2f_1 - 3f_0 \geq 6$ for $g > 0$; see Barnette et al. [1].

3. The high genus case

An interesting extremal case to study is when we fix the number $n := f_0$, and ask for surfaces with the maximal genus, or equivalently, for surfaces with the maximal number of edges and 2-faces. For this we may assume that the surface is triangulated, so $2f_1 = 3f_2$, and $g = 1 + \frac{1}{2}(\frac{f_1}{3} - n)$.

In the combinatorial model, the inequality $f_1 \leq \binom{n}{2}$ is tight for infinitely many values of n, for example for $n = 4, 7, 12$ and for $n \equiv 7 \mod 12$, according to Ringel et al., see [11].

On the other hand, geometric surfaces with $f_1 = \binom{n}{2}$ exist in $\mathbb{R}^3$ for $n = 4, 7$, but *not* for $n = 12$, according to Bokowski & Guedes de Oliveira [2] and Schewe (personal communication, 2005). It is an open problem whether the upper bound $f_1 = O(n^2)$ is tight — the best known lower bound is $f_1 = \Omega(n \log n)$ for surfaces, and $f_1 = \Omega(n^{3/2})$ for the weaker model of "almost disjoint triangles" [7].

4. A combinatorial construction

A combinatorial Ansatz for the construction of extremal surfaces traces back to Brehm [3], see also Datta [5]: In a (p, q)-surface all vertex degrees are p and all faces are q-gons. The goal is to construct (p, p)-surfaces with few vertices.

The Ansatz now produces such surfaces on the vertex set $\mathbb{Z}_N$, by taking as the vertex sets of its faces the cyclic translates of a set $0, A_1, A_2, \ldots A_{2m}$ with successive differences

$$a_1, a_1, a_2, a_2, \ldots, a_m, a_m.$$

This yields a pseudomanifold (and usually a surface) if the a_i are distinct, and the surface will be orientable if the a_i are odd. The key condition to look at is the intersection property, which mandates that the consecutive partial sums of

$$N = A_{2m} \ = \ a_1 + a_1 + a_2 + a_2 + \ldots + a_m + a_m$$

(other than the singleton sums) should be distinct. Datta suggests the choice $a_i = 3^i$, which clearly works, but it is also easy to see that there are choices such as $a_i := m^2 + i - 1$ that yield a sum f_0 of order $O(m^3)$.

The open problem posed in my talk is whether sum of the a_i can be achieved to be $A_{2m} = O(m^2)$, which would clearly be optimal. This asks for a construction of such numbers a_i resp. A_i such that all the differences $A_i - A_j$ are distinct, except for $A_{2k-1} - A_{2k-2} = A_{2k} - A_{2k-1} = a_k$. This asks for a variant of so-called *Sidon sets*. See O'Bryant [10] for a recent survey.

5. A geometric construction

In the last part of the talk, I described a geometric construction that realizes geometric $(p, 2q)$-surfaces in the boundary complex of a polytope of dimension $2 + p(q - 1)$, and hence (after a generic projection) in $\mathbb{R}^5$.

For this, we use an *iterated wedge polytope*, a simple $2 + p(q - 1)$-polytope $W := \Delta_{q-1} \lhd C_p$ with pq facets. We do not describe this polytope here; it arises from a p-gon by p generalized wedge operations, as described in McMullen [9].

The dual polytope $S := \Delta_{q-1} \wr C_p$ is a simplicial $2 + p(q-1)$-polytope that arises from a p-gon by successively replacing each vertex by q new vertices that span a $(q-1)$-simplex, increasing the dimension by $q-1$, with the given vertex in its barycenter. The *wreath product* polytopes were described by Joswig & Lutz [8].

The vertices of W, and hence the facets of S, may be indexed by arrays

$$(k_1, k_2, \ldots, *, *, \ldots, k_p),$$

with $k_i \in [q]$ and two cyclicly adjacent $*$s. Thus there are $f_0 = p\,q^{p-2}$ vertices. The edges of the surface correspond to the arrays of the form

$$(k_1, k_2, \ldots, *, ., \ldots, k_p),$$

with only one $*$, which yields $f_1 = p\,q^{p-1}$ edges. Finally, the faces of the surfaces are p-gons given by those arrays of the form

$$(k_1, k_2, \ldots, ., ., \ldots, k_p),$$

that additionally satisfy the condition $\sum_{i=1}^{p} k_i \equiv 0$ or $1 \pmod q$. This yields a count of $f_2 = 2\,q^{p-1}$ for the faces of the geometric $(p, 2q)$-surface in question.

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