Projected polytopes, Gale diagrams, and polyhedral surfaces

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We report about a new construction scheme that yields interesting 4-dimensional polytopes and polyhedral surfaces. The basic pattern is as follows:

1. Fix a combinatorial type of a high-dimensional simple polytope whose 2-skeleton contains a high-genus surface.
2. Construct explicit matrices for a special "deformed" realization of the polytope.
3. Project it to $\mathbb{R}^4$ such that the graph, as well as the surface realized in the boundary complex, "survive" the projection.
4. Determine the combinatorics (in particular, the f-vector) of the resulting 4-polytope, in terms of Gale duals.
5. Construct a Schlegel diagram, to obtain a polyhedral surface realized in $\mathbb{R}^3$.

In the lecture, we outlined two instances for this program. The first one concerns the "projected products of polygons" presented in [7], whose construction has been simplified and further analyzed by the current authors:

- For $n \geq 4$ even and $r \geq 2$, the product of n-gons $(C_n)^r \subset \mathbb{R}^{2r}$ is a simple $2r$-dimensional polytope with n^r vertices and nr facets.
- A "deformed product realization" P_n^{2r} of $(C_n)^r$ is constructed in terms of explicit, lower block-triangular matrices $A_n^{2r} \in \mathbb{R}^{nr \times 2r}$. (The polytopes P_n^{2r} may be seen as iterated rank 2 deformed products; our construction goes beyond the "rank 1 deformed products" as discussed by Amenta & Ziegler [1].)
- Projection of P_n^{2r} to the last four coordinates yields 4-dimensional polytopes $\pi_4(P_n^{2r})$. All the vertices and edges of P_n^{2r}, as well as all the n-gon 2-faces, are "strictly preserved" by the projection.
- The construction of suitable matrices A_n^{2r}, as well as the combinatorial description of the resulting 4-polytopes $\pi_4(P_n^{2r})$, is achieved in terms of Gale diagrams: The rows of matrices A_n^{2r} are obtained by perturbation of the rows of a reduced matrix $\bar{A}_n^{2r} \in \mathbb{R}^{2n \times 2n}$. Deletion of the last four columns of $\bar{A}^{2r}$ results in a matrix $\bar{\bar{A}}_n^{2r} \in \mathbb{R}^{2n \times 2n-4}$ that is the Gale diagram $G \in \mathbb{R}^{2n \times 3}$ of a pyramid over an $2n$-gon. The rows of A_n^{2r} and their positive dependencies, and thus the faces of $\pi_4(P_n^{2r})$, can be analyzed in terms of lexicographic perturbations of this pyramid.
- The 4-polytopes $\pi_4(P_n^{2r})$ have unusual f-vectors: For $n, r \to \infty$ the *fatness* of these polytopes approaches 9. (This is the largest value currently known. See [5] and [6] for "fatness" and its role for the f-vectors of 4-polytopes.)
- For $n = 4$, $\pi_4(P_n^{2r})$ is a *neighborly cubical polytope*, a cubical 4-polytope with the graph of the $2r$-cube. (These were first obtained by Joswig & Ziegler [2].)
- The Schlegel diagrams of the polytopes $\pi_4(P_n^{2r})$ yield geometric realizations of *equivelar* polyhedral surfaces of type $(4, r)$ in $\mathbb{R}^3$, all of whose faces are quadrilaterals and all of whose vertices have degree r. Thus we have a new construction for a class of surfaces of "unusually high genus" $g \sim N \log N$ on $N = n^r$ vertices, as first obtained by McMullen, Schulz & Wills [4].

A second interesting instance for our construction scheme is as follows:

- For $n \geq 3$, the *totally wedged polytope* W^{2+n} is obtained from an n-gon by forming n successive wedges over facets that correspond to the edges of the original polygon. This is a simple $(2+n)$-dimensional polytope with $2n$ facets. (It is dual to a *wreath product* $C_n \wr I$, as described by Joswig & Lutz [3].)
- The boundary complex of W^{2+n} contains an equivelar surface of type $(4, n)$ consisting of n-gons, where each vertex has degree 4.
- We describe a special "deformed" realization of Q^{2+n} in $\mathbb{R}^{n+2}$ in terms of explicit matrices, designed such that all vertices, edges, and all the n-gon 2-faces "survive" the projection $\pi_4 : \mathbb{R}^{n+2} \to \mathbb{R}^4$ to the last four coordinates.
- The combinatorial structure of $\pi_4(Q^{2+n})$ is analyzed in terms of Gale diagrams.
- Construction of a Schlegel diagram for $\pi_4(Q^{2+n})$ yields equivelar polyhedral surfaces of type $(n, 4)$ in $\mathbb{R}^3$. (This is another family of surfaces of high genus, first constructed by McMullen, Schulz & Wills [4].)

References

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