A Cubical 4-Polytope with an Odd Number of Facets

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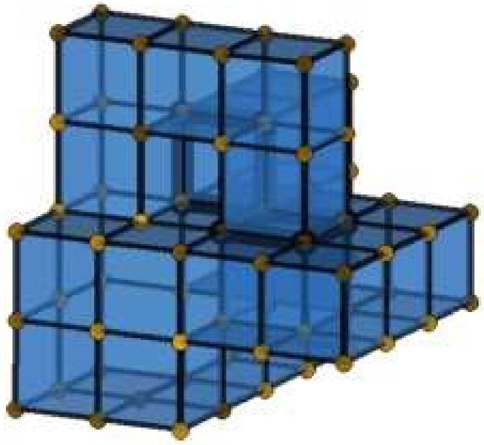

Authors

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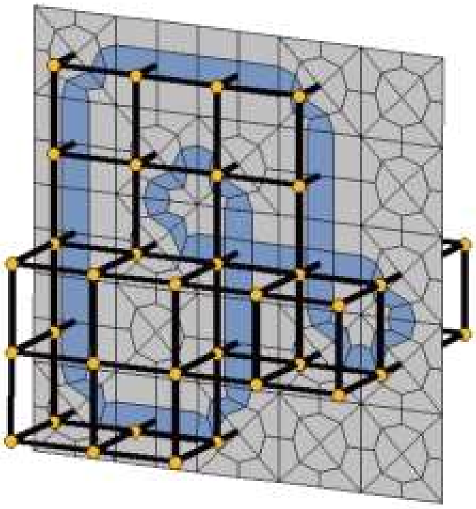
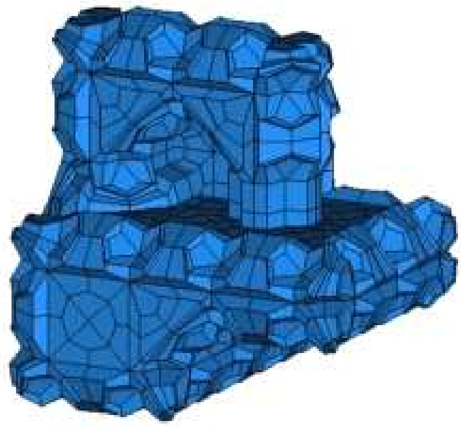
Description

A cubical 4-polytope with an odd number of facets and a dual Boy's surface.

It has been observed by Stanley and by MacPherson that every cubical d-polytope determines a PL immersion of an abstract cubical (d-2)-manifold into (the barycentric subdivision of) the boundary of the polytope [1]. In the case of cubical 4-polytopes each connected component of the dual manifold is a surface (a compact 2-manifold without boundary).



We prove in [3] that every normal crossing codimension one immersion of a compact 2-manifold into R^3 is PL-equivalent to a dual manifold immersion of a cubical 4-polytope. Thus, in particular non-orientable dual 2-manifolds do arise: In [4] we described a rather small and simple instance of a 4-polytope with a non-orientable dual 2-manifold (one component is a Klein bottle). If the dual two-manifold has odd Euler characteristic (with an odd number of non-orientable components of odd genus), the resulting cubical 4-polytope has an odd number of facets.



Here we present an example of a cubical 4-polytope with an odd number of facets, which has Boy surface (an immersed projective plane with a single triple point) as a dual manifold immersion. This solves problems of Eppstein, Thurston and others [2]. Our explicit example has 17,718 vertices and 16,533 facets. From this example, it follows in particular that every combinatorial cube has a subdivision into an even number of cubes (without subdividing the boundary). Moreover, it yields that for any geometric hexa mesh the flip graph (see Bern et al. [2]) has at least two connected components.

Our figures represents the key ideas for the construction: For a given, small (74 vertices) lattice version of Boy's surface, one produces a regular subdivision for a "pile of cubes" that has a subdivision of the Boy surface as a dual manifold. The result (file `C4P_Boy_C3B_with_dual_Boy_surface.hexas`) is then lifted to a 4-polytope by a prism-type construction. (See [3] for details.)

The existence of cubical d-polytopes with an odd number of facets is governed by subtle topological data/obstructions. In particular, such polytopes exist for d=3 (easy) and d=4 (presented here), but not for d=6,8,9 or 10. The cases d=5 and d=7 remain open, for now.

Our submission includes the regular subdivision for a "pile of cubes" that has a subdivision of the Boy surface as a dual manifold (file `C4P_Boy_C3B_with_dual_Boy_surface.hexas`). Furthermore, we provide our set of "templates" used to construct this regular cubical 3-ball: For each possible type of a vertex star of a normal-crossing grid immersion of a surface (compare the image files "`template_*_vertexstar.jpg`"), the set of templates contains a regular cubical subdivision of the standard cube with a dual manifold PL-isomorphic to the vertex star of the grid immersion. This set of templates can be used to produce cubical 4-polytopes with prescribed dual manifold immersion [3, Thm. 7.3]. (Consider the `C4P_Boy_readme.txt` for a description of the file format.)

The cubicality of the 4-polytope `C4P_Boy_Master.poly` can be verified using polymake 2.1 and polymake rules file `c4p.rules`.

Model produced with: polymake 2.1

Keywords	cubical complexes; cubical polytopes; regular subdivision; normal crossing codimension one PL immersion; Boy's surface; grid immersion
MSC-2000 Classification	52B12 (52B11, 52B05)
Zentralblatt No.	

References

1. Eric K. Babson and Clara S. Chan: Counting faces for cubical spheres modulo two, Discrete Math. 212, 3 (2000), 169-183.
2. Marshall Wayne Bern, David Eppstein, and Jeffrey Gordon Erickson: Flipping cubical meshes, Engineering with Computers 18, 3 (2002), 173-187.
3. Alexander Schwartz and Günter M. Ziegler:Construction techniques for cubical complexes, odd cubical 4-polytopes, and prescribed dual manifolds, Experimental Mathematics 13, 4 (2004), 385-413.
4. Alexander Schwartz and Günter M. Ziegler:A cubical 4-polytope with a dual Klein bottle (2004), Electronic Geometry Model No. 2004.05.001, http://www.eg-models.de/2004.05.001.

Files

- XML file: C4P_Boy.xml
- Master File: C4P_Boy_Master.poly
- Applet File: C4P_Boy_Fig1_Applet.jvx
- Applet File: C4P_Boy_Fig2_Applet.jvx
- Applet File: C4P_Boy_Fig3_Applet.jvx
- Preview: C4P_Boy_Fig1_Preview.jpg
- Preview: C4P_Boy_Fig2_Preview.jpg
- Preview: C4P_Boy_Fig3_Preview.jpg
- Readme File: C4P_Boy_readme.txt
- Other: C4P_Boy_C3B_with_dual_Boy_surface.hexas
- Other: template_empty_cubification.hexas
- Other: template_single3_cubification.hexas
- Other: template_single3_vertexstar.jpg
- Other: template_single4a_cubification.hexas
- Other: template_single4a_vertexstar.jpg
- Other: template_single4b_cubification.hexas
- Other: template_single4b_vertexstar.jpg
- Other: template_single5_cubification.hexas
- Other: template_single5_vertexstar.jpg
- Other: template_double8a_cubification.hexas
- Other: template_double8a_vertexstar.jpg

- Other: template_double8b_cubification.hexas
- Other: template_double8b_vertexstar.jpg
- Other: template_triple_cubification.hexas
- Other: template_triple_vertexstar.jpg
- Other: c4p.rules

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