Coloring Hamming graphs, Optimal Binary Codes, and the 0/1-Borsuk Problem in Low Dimensions

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Abstract. The 0/1-Borsuk problem asks whether every subset of $\{0,1\}^d$ can be partitioned into at most $d+1$ sets of smaller diameter. This is known to be false in high dimensions (in particular for $d \geq 561$, due to Kahn & Kalai, Nilli, and Raigorodskii), and yields the known counterexamples to Borsuk's problem posed in 1933.
Here we ask whether there might be counterexamples in low dimension as well. We show that there is no counterexample to the 0/1-Borsuk conjecture in dimensions $d \leq 9$. (In contrast, the general Borsuk conjecture is open even for $d = 4$.)
Our study relates the 0/1-case of Borsuk's problem to the coloring problem for the Hamming graphs, to the geometry of a Hamming code, as well as to some upper bounds for the sizes of binary codes.

1 Introduction

The Borsuk conjecture is a puzzling problem: posed in 1933, in the famous paper by K. Borsuk [6] that contained the "Borsuk-Ulam theorem," it has resisted all attempts of proof until in 1992 Kahn and Kalai [11] announced that the conjecture is false, due to counterexamples in dimensions 1325 and higher. After much subsequent work, we now know that the Borsuk conjecture is false in all dimensions $d \geq 560$, and true in dimensions $d \leq 3$ — which leaves a remarkable gap! How about dimension 4, say? This leads us to ask: "*Must the counterexamples be necessarily be so high-dimensional?*"

It turns out that while the proofs in dimensions $d \leq 3$ depend on intricate geometric arguments, all the counterexamples rely on purely combinatorial work on sets of 0/1-vectors plus some linear algebra techniques. Thus we ask: "*Must 0/1-counterexamples be necessarily be so high-dimensional?*" This question leads us to a lot of beautiful combinatorics, to graph coloring problems and optimal codes, and finally to a partial answer: Perhaps they don't have to be *that* high-dimensional, but at least there are no counterexamples in dimensions $d \leq 9$.

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Acknowledgements. Please consider this paper a survey: There is very little new in it, it's just making connections between different facts, some of which are published, and some of which seem to be "well-known" (that is, well-known to those who well know them). I am reporting also about some contributions that I was shown by Lex Schrijver, Stefan Hougardy, Jon McCammond, and Noga Alon — thanks to them! I am also grateful to my 'Diplomanden' Jürgen Petersen [18] and Frank Schiller [22] for their contributions, and for many discussions.

2 The 0/1-Borsuk Problem

Borsuk asked:

Borsuk's problem: Is it true that every bounded subset S of $\mathbb{R}^d$ can be decomposed into $d+1$ subsets,

$$S = S_1 \cup S_2 \cup \ldots \cup S_{d+1},$$

all of which have smaller diameter than S?

The number of $d+1$ subsets cannot be reduced: $d+1$ sets are needed, for example, if S is a regular simplex of dimension d (or just its vertex set), or a d-dimensional ball (or just its boundary). In both cases, however, a partition into $d+1$ parts exists, and isn't hard to find. (Only the part that a d-ball cannot be partitioned into fewer than $d+1$ parts of smaller diameter is non-trivial; it is equivalent to the Borsuk-Ulam theorem, and was anticipated by Lyusternik and Shnirel'man in 1930, three years before Borsuk's paper.)

Borsuk's conjecture was proved to be true for all sets S of dimension $d \leq 3$ (Perkal 1947; Eggleston 1955), and for smooth bodies S (Hadwiger 1946), but the general case remained an open problem for a long time. See Grünbaum [8] and Boltyanski, Martini & Soltan [5, §31] for surveys of Borsuk's problem. On the other end, the constructions of Kahn & Kalai were simplified, extended and improved, so that with the efforts of Nilli [16], Raigorodskii [20, 21] and Weißbach [24] we have counterexamples for all $d \geq 560$. (See [1] for a popular exposition.) Thus, with all the work and effort that was put into the problem, we know now that the answer is "yes" for $d \leq 3$ and "no" for $d \geq 560$.

Borsuk's problem is hard enough for the special case where S is a finite set (equivalently, if one considers convex polytopes $\mathrm{conv}(S)$, since the largest distance in a polytope always occurs between two vertices). It is an interesting question whether one can derive the general case from the polytope case.

An even more special case (but the one used to construct counterexamples!) is the one when S is a set of 0/1-vectors, that is, where $S \subseteq \{0,1\}^d$ is a subset of the vertices of the regular 0/1-cube. In that special case, it is now known that Borsuk's conjecture is false for $d \geq 561$, but true for $d \leq 9$. For the counterexamples in high dimensions, we refer to the sources quoted above; our aim in the following is to demonstrate the positive answer for the "0/1-Borsuk problem" for $d \leq 8$, and to explore some of the combinatorics, graph theory and coding theory connected with it.

0/1-Borsuk problem: For which d can every subset $S \subseteq \{0,1\}^d$ be decomposed into $d+1$ subsets,

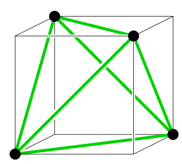
$$S = S_1 \cup S_2 \cup \ldots \cup S_{d+1},$$

all of which have smaller diameter than S?

It is not true that $d+1$ subsets are even needed in each dimension: For example, in dimension 2 it is easy to check that 2 subsets will always do. However, the upper bound cannot be improved much.

Example 1. The subset $S := \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ is the vertex set of a regular $(d-1)$-dimensional simplex of edge length $\sqrt{2}$. This set can be decomposed into d subsets of smaller diameter, but not into fewer.

Example 2. A regular d-simplex with vertices in $\{0,1\}^d$ exists if and only if there is a Hadamard matrix of order $d+1$ (see [25]). For this $d+1$ is necessarily equal to 1, to 2, or to a multiple of 4 (and it is conjectured that Hadamard matrices exist for all multiples of 4). Thus in dimensions d such that a Hadamard matrix of order $d+1$ exists, we have an example of a subset S which needs $d+1$ parts for its decomposition. Our figure displays a corresponding set for $d = 3$.



In the following, we'll treat the 0/1-Borsuk problem "case by case" in terms of two parameters; the first one is the dimension d, the second one is the integer k, with $1 \le k \le d$, such that $\sqrt{k}$ is the diameter of the set $S \subseteq \{0,1\}^d$ that we consider. Equivalently – and this will be useful in a coding theory context – the parameter just denotes the ℓ_1-diameter, or Hamming diameter, of the set: for two 0/1-vectors $\boldsymbol{x}, \boldsymbol{y}$, the distance $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{k}$ is given by the number $k = \|\boldsymbol{x} - \boldsymbol{y}\|_1$ of coordinates in which $\boldsymbol{x}$ and $\boldsymbol{y}$ differ (by ± 1).

Thus, for every $d \ge 1$ and $1 \le k \le d$, we'll be studying the following problem:

Borsuk(d,k): Is it true that every subset $S \subseteq \{0,1\}^d$ of diameter $\sqrt{k}$ can be decomposed into $d+1$ subsets,

$$S = S_1 \cup S_2 \cup \ldots \cup S_{d+1},$$

all of which have smaller diameter than S?

In this formulation, the 0/1-Borsuk problem is true in dimension d if and only if Borsuk(d,k) is true for all $k \in \{1, \ldots, d\}$.

At this point, it is an instructive exercise to work out the 0/1-Borsuk problem for $d \le 3$ – an exercise, however, that we leave to the interested reader.

3 Reformulation as a Coloring Problem

Assume that we are handed a particularly interesting set $S \subseteq \{0,1\}^d$ of diameter $\sqrt{k}$, or just an example for which we are *told* that it is particularly interesting, and we are asked whether it satisfies Borsuk's conjecture, what should we do?

It seems that the problem is difficult, just because it is equivalent to a coloring problem, and coloring problems are difficult (in general), and we have not much indication that this one happens to be an easy special case.

Definition 1. We shall say that a subset $S \subseteq \{0,1\}^d$ of (Euclidean) diameter $\sqrt{k}$ has *Hamming diameter* k. It's *Borsuk graph* is the graph $B_k(S)$ with vertex set S, and with an edge between vertices $\boldsymbol{x}, \boldsymbol{y} \in S$ if the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$ is the diameter of S (that is, if the Euclidean distance is $\sqrt{k}$, and the Hamming distance is k).

Now a partition of S into m subsets of smaller diameter is just the same thing as a partition of the vertex set of $B_k(S)$ into m stable subsets, that is, a coloring the graph $B_k(S)$ with m colors. This leads to the following "coloring version" of the 0/1-Borsuk problem. (This reduction to a coloring problem can be done the same way for the case of a general set $S \subseteq \mathbb{R}^d$, but it fails in the infinite case.)

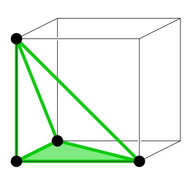
Borsuk(d,k): Is it true that for every subset $S \subseteq \{0,1\}^d$ of Hamming diameter k, the corresponding Borsuk graph has chromatic number

$$\chi(B_k(S)) \quad \leq \quad d+1\,?$$

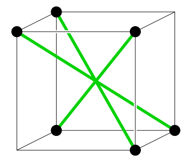
In this formulation, let's discuss (and get rid of) a number of simple cases:

$k = 1$: If $k = 1$, then $|S| = 2$, $B_1(S) \cong K_2$, $\chi(B_1(S)) = 2 \leq d+1$. No problem.

$k = 2$: 0/1-configurations of Hamming diameter 2 are easily classified, modulo 0/1-*equivalence* (cf. [25]), where we consider two sets as equivalent if we can transform one into the other using just symmetries of the 0/1-cube, that is, by permutation and complementation of coordinates.
The result is that there is one special case: the 3-dimensional regular tetrahedron discussed and depicted above, whose Borsuk graph is a K_4, and thus the chromatic number is $\chi(B_2(S)) = \chi(K_4) = 4 \leq d+1$. Any configuration that is not 0/1-equivalent to this one is 0/1-equivalent to a subset to the configuration $S = \{0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$, whose Borsuk graph is $K_d + K_1$, a clique plus an isolated vertex; we get for this case that $\chi(B_2(S)) \leq d$. Thus Borsuk(d,k) holds for $k \leq 2$.



***k* is odd:** In this case $B_k(S)$ is bipartite; we get a legal coloring by collecting the 0/1-vectors with odd coordinate sums, and the vertices with even coordinate sums, into two distinct classes. Thus here we have $\chi(B_k(S)) = 2$. Our figure shows such a case, where the points in S are the vertices of a 0/1-octahedron.



Thus we have dealt with the cases $k \leq 3$; the case $k = 4$ is already harder, see Section 9.2. Our next goal will be the situation where k is rather large (compared with d); it turns out that the bounds we get there are more generally valid for the "Hamming graphs," which contain the Borsuk graphs as subgraphs.

4 Coloring Hamming Graphs

There are lots of different Borsuk graphs for any given d and k: but they are all subgraphs of the following "Hamming graphs."

Definition 2. The *Hamming graph* $H_{d,k}$ has vertex set $\{0,1\}^d$, and two of its vertices are connected by an edge if and only if they have Hamming distance exactly k (that is, Euclidean distance $\sqrt{k}$.)

(We could have also used the notation $B_k(\{0,1\}^d)$ for $H_{d,k}$, but our convention is that the subscript k in $B_k(S)$ should denote the diameter of S.)

Lemma 1. *For every* 0/1*-set* $S \subseteq \{0,1\}^d$ *of diameter* k *we have*

$$\chi(B_k(S)) \quad \leq \quad \chi(H_{d,k}).$$

Proof. The Borsuk graph $B_k(S)$ is an induced subgraph of $H_{d,k}$. □

In particular, this means that we have proved Borsuk(d,k) if we find out that $\chi(H_{d,k}) \leq d+1$. This holds for some parameters, while it is drastically false for others, as we shall see – it fails even for some d with $k = 2$, where we have already established that Borsuk(d,k) is correct.

The following is a look at the Hamming graphs (and their chromatic numbers) for special parameters and examples.

***k* is odd:** In this case $H_{d,k}$ is bipartite, and thus $\chi(H_{d,k}) = 2$, with the same argument as used above for the Borsuk graphs.

***k* is even:** In this case a vertex with odd coordinate sum, and a vertex with even coordinate sum, cannot be connected by an edge. Thus the Hamming graph $H_{d,k}$ is disconnected for even k. The two components induced on the even vertices and on the odd vertices are isomorphic, and can be identified with the graph

$$H_{d-1,\{k,k-1\}} \quad = \quad H_{d-1,k} \cup H_{d-1,k-1},$$

two of whose vertices are connected by an edge if their Hamming distance is either k or $k-1$. In particular, for $k=2$ the graph $H_{d-1,\{k,k-1\}}$ has an edge between any two distinct vertices of distance *at most* 2, and may be denoted by $H_{d-1,\leq 2}$.

k is large: If vertices $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^d$ have distance k, then this means that they differ in k coordinates, and thus the first $d-k+1$ coordinates of $\boldsymbol{x}$ and of $\boldsymbol{y}$ cannot be all the same. This implies that

$$\boldsymbol{x} \quad \longmapsto \quad (x_1, \ldots, x_{d-k+1}) \;\in\; \{0,1\}^{d-k+1}$$

is a legal coloring of $H_{d,k}$. The existence of this coloring implies that

$$\chi(H_{d,k}) \quad \leq \quad 2^{d-k+1}.$$

This bound is meaningful if $d-k$ is small; it is sharp if $d-k$ is very small. (It corresponds to the “Singleton bound” in coding theory.)

$k = d$: As a special case for the Singleton Bound, for $d=k$ we see that $H_{d,d}$ is a matching (of chromatic number 2).

$k = d-1$: The Singleton bound implies that the chromatic number of $H_{d,d-1}$ is at most 4. On the other hand, it is easy to see that for even $d-1$ the graph $H_{d,d-1}$ is not bipartite. Furthermore, a result of Payan [17] states that a “cubelike graph” such as $H_{d,d-1}$ cannot have chromatic number 3, so we find that $\chi(H_{d,d-1}) = 4$ for odd d, and $\chi(H_{d,d-1}) = 2$ for even d.

5 Some Coding Theory Bounds are Used

The Hamming graphs $H_{d,2}$ have been studied extensively, also since they are unions of two components that are isomorphic to $H_{d-1,\leq 2}$. See for example [12] and [7], and especially the discussion in [10, Sect. 9.7] (and the references quoted there).

Let’s discuss upper and lower bounds for the chromatic numbers of these graphs independently.

Lemma 2. (Linial, Meshulam & Tarsi [12]) *For all* $d \geq 1$,

$$\chi(H_{d,2}) \quad \leq \quad 2^{\lceil \log_2(d) \rceil},$$

where the upper bound can be read as “d rounded up to the next power of 2.*”*

Proof. Let $d \leq 2^m$, then an explicit 2^m-coloring can be given as follows. For each coordinate i $(1 \leq i \leq d)$, let $b(i-1) \in \{0,1\}^m$ be the 0/1-vector of length m

obtained from the binary expansion of $i-1$, adding leading zeroes as necessary. Then we color the vertices of $H_{d,2}$ by

$$\begin{aligned} c : \{0,1\}^d &\longrightarrow \{0,1\}^m, \\ \boldsymbol{x} &\longmapsto \sum_{i\,:\,x_i=1} b(i-1), \end{aligned}$$

where the sum is taken "component-wise," modulo 2. Thus if two vectors $\boldsymbol{x}, \boldsymbol{y} \in \{0,1\}^d$ differ in exactly two coordinates i, j, then their colors $c(\boldsymbol{x})$ and $c(\boldsymbol{y})$ will differ exactly by $b(i-1)+b(j-1)$ (modulo 2), which is not zero since $i \neq j$. □

According to MacWilliams & Sloane [14, Part II, p. 523], "Probably the most basic problem in coding theory is to find the largest code of a given length and minimum distance," that is, the evaluation and estimation of the quantities given by the following definition.

Definition 3. For $d \geq 1$ and $1 \leq s \leq d$, $A(d,s)$ denotes the maximum number of codewords in any binary code of length d and minimum distance s between the codewords.
That is, $A(d,s)$ is the largest size of a subset $C \subseteq \{0,1\}^d$ such that any two elements of C have Hamming distance at least s.

We refer to MacWilliams & Sloane [14, Chap. 17] for non-trivialities about these quantities, and their relevance. An excellent source for the "linear programming bounds" that are used to get non-trivial upper bounds (such as the ones used below) is [3]. As an example, we trivially have $A(d,1) = 2^d$, and $A(d,d) = 2$.

Lemma 3. *For all* $t \geq 1$ *and* $d \geq 2t$, $\quad A(d,2t) = A(d-1,2t-1)$.

Proof. Indeed, if we take any $(d,2t)$-code, then the operation of "deleting the last coordinate" yields a code of the same size (by $t \geq 1$) and minimum distance decreased by at most 1, and "adding a parity check-bit as a last coordinate" will take us from a $(d-1,2t-1)$-code to a $(d,2t-1)$-code of the same size in which furthermore all code words and hence all distances are even, and which is thus a $(d,2t)$-code. □

In particular, $A(d,4) = A(d-1,3)$ holds for all $d \geq 4$. Now $A(d-1,3)$ is "by definition" also the largest size of an independent ("stable") set in the graph $H_{d-1,\leq 2}$. Together with the fact that $H_{d,2}$ consists of two components that are isomorphic to $H_{d-1,\leq 2}$, this yields that the largest size of an independent set in $H_{d,2}$ is exactly $2\,A(d,4)$.

At this point, we quote a result by Best & Brouwer [4] [3, p. 129] about shortened Hamming codes, which implies that

$$A(2^m - t, 4) \quad = \quad 2^{2^m - t - m - 1} \qquad \text{for } \ 0 \leq t \leq 3.$$

This result, translated back to graph theory, says that the independent sets in the Hamming graphs $H_{d,2} = A(d,4)$ "aren't that large." Thus it provides a lower bound on the chromatic numbers via the inequality

$$\chi(G) \quad \geq \quad \frac{|V|}{\alpha(G)}$$

(where $\alpha(G)$ denotes the size of a largest independent set of vertices in G), which is valid for every finite graph G. Applied to $G = H_{d,2}$ for $d = 2^m - t$, this yields

$$\chi(H_{2^m-t,2}) \quad \geq \quad \frac{2^{2^m-t}}{2\ 2^{2^m-t-m-1}} \quad = \quad 2^m.$$

Thus we get the following result, which says that the upper bound of Lemma 2 is sharp for *some* values of d.

Proposition 1. (Linial, Meshulam & Tarsi [12]) *For all* $d \geq 1$*,*

$$\chi(H_{d,2}) \quad \leq \quad 2^{\lceil \log_2(d) \rceil},$$

with equality if d is of the form $d = 2^m$, $2^m - 1$, $2^m - 2$, or $2^m - 3$.

In particular, $\chi(H_{d,2}) = 2^{\lceil \log_2(d) \rceil}$ holds for all $d \leq 8$, and again for $13 \leq d \leq 16$. Of course this raises questions for the other values, in particular for $d = 9$. Let's increase the suspense a bit and postpone this question to Section 9.1.

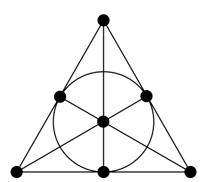
6 A Hamming Code is Used

Payan [17] has proved that $\chi(H_{6,4}) \leq 7$ by exhibiting an explicit coloring. His claim that "it is not very difficult to prove" that the chromatic number is indeed 7 can be confirmed by computer [22]. Thus Payan disproved an earlier conjecture that "cube-like" graphs must necessarily have a chromatic number that is a power of 2.

In the following we will discuss that/why the chromatic numbers of both $H_{7,4}$ and $H_{8,4}$ are 8 — and we want to do that "by hand" (rather than leave it to a computer) since the geometry of the argument is so nice (due to Lex Schrijver [23]).

Lemma 4. $\chi(H_{7,4}) \ = \ \chi(H_{8,4}) \ = \ 8.$

Proof. We start with the binary Hamming code $H(3) \subseteq \{0,1\}^7$, which can be described as follows: Number the points of a Fano plane



by $1, 2, \ldots, 7$, and take as the code words of your code the zero vector (corresponding to the empty subset), the all-ones vector (corresponding to the whole plane), the seven 0/1-vectors of weight 3 that correspond to the lines of the Fano plane, and the seven 0/1-vectors of weight 4 that correspond to their complements. Altogether this yields the 16 codewords of the Hamming code $H(3)$, about which the following facts are well-known and easily verified:

- $H(3)$ is a linear code (sums of codewords modulo 2 are codewords),
- it is a perfect code of minimum distance 3 (that is, every 0/1-vector of length 7 is either a code word, or it has distance 1 from a codeword),
- the complements of codewords are codewords as well.

Now the eight even code words, of weights 0 and 4 (corresponding to the empty set and to the complements of the lines in the Fano plane), all have distance 4 from each other, and thus we have found an 8-clique in $H_{7,4}$. At the same time, we can describe an 8-coloring of $H_{7,4}$ associated with this clique: for this we take eight colors for the eight code words in the clique, and give the same color to all the 0/1 vectors that have distance at most 1 either from the code word or from its complement. In other words, all vectors of distances $0, 1, 6$ or 7 from an even code word get the same color, which yields a perfect 8-coloring.

To treat $H_{8,4}$, we use the extended Hamming code $\widetilde{H}(3) \subseteq \{0,1\}^8$, which is obtained by extending the code words of $H(3)$ by parity check bits. The resulting code is a linear code consisting of 16 code words, of minimum distance 4: indeed, all words other than the zero word and the all-ones word have weight 4. (Geometrically, this code corresponds to a remarkable 8-dimensional regular cross polytope of edge length $\sqrt{4} = 2$, whose vertices form a subset of the 0/1-cube.) Now the clique in $H_{7,4}$ consisted of the even code words, so it also determines an 8-clique in $H_{8,4}$. But it also yields an 8-coloring: for this we give the same color to all the 0/1-vectors that extend a vector of distance $0, 1, 6$ or 7 from an even code word in $H(3)$. It is easily checked that no two vectors of Hamming distance 4 are assigned the same color by this rule. □

7 Coloring Hamming Graphs, III

Let's collect the information that we have for the chromatic numbers of $H_{d,k}$ for $d \leq 9$ in a table:

$d \setminus k$	1	2	3	4	5	6	7	8	9
1	2								
2	2	2							
3	2	4	2						
4	2	4	2	2					
5	2	8	2	4	2				
6	2	8	2	7	2	2			
7	2	8	2	8	2	4	2		
8	2	8	2	8	2	≤ 8	2	2	
9	2	≤ 16	2		2	≤ 16	2	4	2

8 The 0/1-Borsuk problem in low dimensions

The following table combines the upper bounds achieved in Section 3 with our knowledge about the chromatic numbers of Hamming graphs, which we have just summarized. The entry for d and k gives the best upper bound available for the maximal chromatic number of a Borsuk graph $B_k(S)$ for a subset $S \subseteq \{0,1\}^d$ of Hamming diameter k:

$d \setminus k$	1	2	3	4	5	6	7	8	9
1	2								
2	2	2							
3	2	4	2						
4	2	4	2	2					
5	2	5	2	4	2				
6	2	6	2	7	2	2			
7	2	7	2	8	2	4	2		
8	2	8	2	8	2	8	2	2	
9	2	9	2		2		2	4	2

Theorem 1. *The* 0/1*-Borsuk problem has a "yes" answer for all* $d \leq 9$*.*

Proof. For $d \leq 8$, this follows from the fact that for $d \leq 8$ all the entries in the above table are at most $d+1$.

The case $d = 9$ was done by Petersen [18], and relies on explicit coloring schemes that are rather complicated and will thus not be described or verified here. □

9 Coloring Hamming Graphs, III

9.1 The Hamming graphs $H_{d,2}$ – Stefan Hougardy

We had postponed the (interesting) case $d \geq 9$ of the graphs $H_{d,2}$. In particular, for $d = 9$ we can use that $A(8,3) = A(9,4) = 20$ [3], which yields a lower bound of $\lceil \frac{2^9}{2 \cdot 20} \rceil = 13$ for the chromatic number of $H_{9,2}$.

On the other hand, several of the better coloring heuristics do find a 14-coloring of $H_{9,2}$. Apparently this was first done by Hougardy in 1991, who got it from an adaption of the Petford-Welsh [19] algorithm. This leaves us with a rather narrow gap

$$13 \quad \leq \quad \chi(H_{9,2}) \quad \leq \quad 14.$$

The question whether the chromatic number of $H_{9,2}$ is 13 or 14 is a combinatorial covering problem: We are trying to cover the $2^8 = 256$ even vertices in the 9-cube by even $(9,4)$-codes. Hougardy [9] has found that there are only two non-equivalent such codes of the maximal size 20, but many more of sizes 19 or 18.

Now if a covering by 13 codes exist, then it must use at least 9 codes of size 20. Do they fit together?

More generally, if we want to get beyond the basic $\frac{|V|}{\alpha(G)} \leq \chi(G)$ lower bound for the chromatic number, then we must find out more about the "geometry" of the independent sets: these might be large enough, but they might not fit together to give a coloring with few colors.

For the higher values of $d = 10, 11, 12$, the available data are

$$\begin{array}{rcrcl rcccl} & & A(9,4) & = & 20 & 13 & \leq & \chi(H_{9,2}) & \leq & 14 \\ & & A(10,4) & = & 40 & 13 & \leq & \chi(H_{10,2}) & \leq & 16 \\ 72 & \leq & A(11,4) & \leq & 76 & 14 & \leq & \chi(H_{11,2}) & \leq & 16 \\ 144 & \leq & A(12,4) & \leq & 152 & 14 & \leq & \chi(H_{12,2}) & \leq & 16 \\ & & A(13,4) & = & 256 & & & \chi(H_{13,2}) & = & 16 \end{array}$$

The upper bounds for $A(11,4)$ and $A(12,4)$ are due to Litsyn & Vardy [13]; it is conjectured, however, that the lower bounds are tight [3, p. 128], which in turn would give a lower bound of 15 for $\chi(H_{11,2})$ and $\chi(H_{12,2})$. But a gap remains in either case ...

9.2 A lower bound for small diameter – Jon McCammond

Petersen [18] has shown that Borsuk$(d,4)$ has a "yes" answer for all large enough d. And in fact, McCammond [15] has proved that for every fixed k the answer to Borsuk(d,k) is positive when d is large enough (with respect to k).

Even stronger, it follows from the arguments and bounds obtained in [15] that Borsuk(d,k) is true whenever $k \leq c\sqrt{\log d}$, for some constant $c > 0$. Thus, for counterexamples k can not be too small compared to d.

9.3 An upper bound for large diameter – Noga Alon

One can also show that for counterexamples to Borsuk(d,k) the difference $d-k$ can not be too small when compared to d. For this, the "Singleton bound" that we had used in the case of very small $d-k$ is far from optimal.

Proposition 2. (Alon [2])
If $l \leq d$ is such that a Hadamard matrix $H_\ell \in \{1,-1\}^\ell$ exists, then

$$\chi(H_{d,>d-\sqrt{\ell}}) \quad \leq \quad 2\ell.$$

Proof. For simplicity, we describe this proof for vertex coordinates in $\{1,-1\}$. Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell \in \{1,-1\}^\ell$ be the columns of H_ℓ, which form an orthogonal basis by definition. Thus the points $\{\pm\boldsymbol{v}_1, \ldots, \pm\boldsymbol{v}_\ell\} \subseteq \{1,-1\}^\ell$ form the vertices of a regular cross polytope of edge length $\sqrt{2}\sqrt{\ell}$.

The points in $\{1,-1\}^\ell$ are now 2ℓ-colored according to the closest point from the set $\{\pm\boldsymbol{v}_1, \ldots, \pm\boldsymbol{v}_\ell\}$. (In case of draws, decide arbitrarily.)

If an arbitrary vector $\boldsymbol{x} \in \{1,-1\}^\ell$ is expanded as $\boldsymbol{x} = \sum_{i=1}^{\ell} \lambda_i \boldsymbol{v}_i$ in terms of the orthogonal basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\ell$, then one of the coefficients $\lambda_i = \frac{\langle \boldsymbol{x}, \boldsymbol{v}_i \rangle}{\langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle}$ has absolute value $|\lambda_i| \geq \frac{1}{\sqrt{\ell}}$, that is, $\boldsymbol{x}$ has at least $\frac{1}{2}(\ell + \sqrt{\ell})$ components in common with the corresponding vector $\pm \boldsymbol{v}_i$. Thus two vectors in $\{1,-1\}^\ell$ that get the same color have at least $\sqrt{\ell}$ components in common.

From this we derive that if two vectors $\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}} \in \{1,-1\}^d$ get the same color (according to their first ℓ coordinates), then they agree in at least $\sqrt{\ell}$ coordinates. □

The point set used in this proof corresponds to a well-known structure in the 0/1-cube. Namely, the corresponding set

$$\mathcal{C}_\ell \quad := \{\tfrac{1}{2}(\mathbf{1} \pm \boldsymbol{v}_1), \ldots, \tfrac{1}{2}(\mathbf{1} \pm \boldsymbol{v}_\ell)\} \quad \subseteq \quad \{0,1\}^\ell$$

is a binary code of length ℓ, minimum distance $\frac{1}{2}\ell$, consisting of 2ℓ code words, which form the vertices of a regular cross polytope of edge length $\sqrt{\frac{1}{2}\ell}$. It is known as the $(\ell, 2\ell, \frac{1}{2}\ell)$-Hadamard code [14, Part I, p. 49].

In the special case $\ell = 8$, this Hadamard code is equivalent to the extended Hamming code $\widetilde{H}(3)$ that we had used for a different purpose in Section 6. Thus we could have equivalently started the construction in Section 6 with the 8×8 Hadamard matrix instead of the Fano plane.

Together with the known existence results for Hadamard matrices (they are conjectured to exist for all $\ell = 4k$, and known to exist whenever $4k-1$ is a prime power), Proposition 2 shows that Borsuk(d,k) is true whenever $d - k \leq c'\sqrt{d}$ for a constant $c' > 0$, which for large d can be taken to be arbitrarily close to $\frac{1}{2}\sqrt{2}$.

The counterexamples to Borsuk's conjecture due to Kahn & Kalai and the variations of Nilli, Raigorodski and Weißbach all have have $k \sim \frac{d}{2}$.

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