ON FREE PLANES IN LATTICE BALL PACKINGS

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Abstract

This note, by studying relations between the length of the shortest lattice vectors and the covering minima of a lattice, mainly proves that for every d-dimensional packing lattice of balls one can find a 4-dimensional plane, parallel to a lattice plane, such that the plane meets none of the balls of the packing, provided the dimension d is large enough. On the other hand, we show that for certain ball packing lattices the highest dimension of such "free planes" is far from d.

1. Introduction

Throughout the paper let $\mathbb{R}^d$ be the d-dimensional Euclidean space equipped with the Euclidean norm $\|\mathbf{x}\|$ and the inner product $\langle \mathbf{x}, \mathbf{y}\rangle$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Let B^d be the d-dimensional unit ball centered at the origin $\mathbf{o}$. As usual a lattice $\Lambda \subset \mathbb{R}^d$ is defined as the set of all integral combinations of d linearly independent vectors $\mathbf{b}_i \in \mathbb{R}^d$, i.e.,

$$\Lambda = \oplus_{\mathbb{Z}} \mathbf{b}_i = \{z_1\mathbf{b}_1 + \cdots + z_d\mathbf{b}_d : \ z_i \in \mathbb{Z}\}.$$

Every set of d vectors $\{\tilde{\mathbf{b}}_1, \ldots, \tilde{\mathbf{b}}_d\}$ with $\oplus_{\mathbb{Z}}\tilde{\mathbf{b}}_i = \Lambda$ is called a basis of the lattice. An l-dimensional affine subspace $L \subset \mathbb{R}^d$ is called an l-dimensional *lattice plane* if $\dim(L \cap \Lambda) = l$. For a plane L the orthogonal projection of a set $X \subset \mathbb{R}^d$ onto the orthogonal complement of L is denoted by $X|L^\perp$. For an l-dimensional lattice plane L with respect to a lattice Λ the set $\Lambda|L^\perp$ is a $(d-l)$-dimensional lattice. The length of the shortest non-zero vectors in a lattice Λ is denoted by $\lambda_1(\Lambda)$, i.e.,

$$\lambda_1(\Lambda) = \min\{\|\mathbf{b}\| : \ \mathbf{b} \in \Lambda\backslash\{\mathbf{o}\}\}.$$

$\lambda_1(\Lambda)$ is also called the *homogeneous minimum* of Λ, and as a counterpart we have the *inhomogeneous minimum* $\mu(\Lambda)$ defined by

$$\mu(\Lambda) = \max_{\mathbf{x}\in \mathrm{lin}(\Lambda)} \min\{\|\mathbf{x} - \mathbf{b}\| : \ \mathbf{b} \in \Lambda\}.$$

This means that $\mu(\Lambda)$ is the smallest number μ with the property that $\Lambda + \mu B^d$ covers the space $\mathbb{R}^d$. For this reason, $\mu(\Lambda)$ is also called the *covering radius* of Λ. In [14] the covering radius has been embedded in a series of functionals $\mu_i(\Lambda)$ which, in our setting, can be defined by

$$\mu_i(\Lambda) = \max_L \left\{\mu\left(\Lambda|L^\perp\right)\right\},$$

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where the maximum is taken over all $(d-i)$-dimensional lattice planes L of Λ. $\mu_i(\Lambda)$ is called the i-th covering minima of Λ. In fact, it is the maximal covering radius among all i-dimensional lattices arising by those projections of Λ. Obviously, we have

$$\mu(\Lambda) = \mu_d(\Lambda) \geq \mu_{d-1}(\Lambda) \geq \cdots \geq \mu_1(\Lambda).$$

A lattice $\Lambda \subset \mathbb{R}^d$ is called a *packing lattice* of B^d if, for two different points $\mathbf{x}$ and $\mathbf{y}$ of Λ, the translates $\mathbf{x} + B^d$ and $\mathbf{y} + B^d$ have no interior points in common. In other words, a lattice Λ is a packing lattice of B^d if and only if $\lambda_1(\Lambda) \geq 2$, and the homogeneous minimum can also be defined by

$$\frac{2}{\lambda_1(\Lambda)} = \min\left\{\lambda \in \mathbb{R}_{>0} : \ \lambda\Lambda \text{ is a packing lattice of } B^d\right\}.$$

So we have

$$\mu\left(\frac{2}{\lambda_1(\Lambda)}\Lambda\right) \geq 1 \quad \text{or} \quad \mu_d(\Lambda) \geq \frac{\lambda_1(\Lambda)}{2}$$

for every lattice $\Lambda \subset \mathbb{R}^d$.

By the definition of the covering minima and the homogeneous minimum, the statement "$\mu_i(\Lambda) \geq \frac{\lambda_1(\Lambda)}{2}$ *for all d-dimensional lattices*" has the following equivalent geometrical interpretation: *For every d-dimensional lattice ball packing one can find a $(d-i)$-dimensional plane H, parallel to a lattice palne, which does not meet the balls of the packing in their interior.* One may say that such a plane is "free".

In 1960, by an elementary method Heppes [9] proved that

$$\mu_2(\Lambda) > \frac{\lambda_1(\Lambda)}{2}$$

for every 3-dimensional lattice. In other words, in every 3-dimensional lattice ball packing one can find a cylinder of infinite length which does not intersect any of the balls. Later this result was generalized in [10] and [13] to high dimensions, i.e., for $d \geq 3$ one has

$$\mu_{d-1}(\Lambda) > \frac{\lambda_1(\Lambda)}{2}.$$

As a counterpart it was shown by Hausel [7], correcting a claim of [11], that there exists a constant α such that in every dimension d there exists a lattice $\tilde{\Lambda} \subset \mathbb{R}^d$ with

$$\mu_{\lfloor \alpha\sqrt{d} \rfloor}(\tilde{\Lambda}) < \frac{\lambda_1(\tilde{\Lambda})}{2}.$$

This note improves both results by the following theorems:

Theorem 1.1. *Let $\Lambda \subset \mathbb{R}^d$ be a lattice. We have*

$$\mu_{d-k}(\Lambda) > \frac{\lambda_1(\Lambda)}{2}$$

in the cases i) $d \geq 5$ *and* $k = 2$, ii) $d = 8$ *and* $k = 3$, iii) d *sufficiently large and* $k = 4$.

Hence for sufficiently large d, for every d-dimensional lattice ball packing we can find a 4-dimensional plane H which does not meet any of the balls of the packing. On the other hand, we will prove

Theorem 1.2. *There exists a constant $\beta < 1$ such that in every dimension one can find a lattice $\tilde{\Lambda}$ with*

$$\mu_k(\tilde{\Lambda}) < \frac{\lambda_1(\tilde{\Lambda})}{2}, \qquad k \le \beta d.$$

So in general we can not find a "free" plane of $(1-\beta)d$-dimensions with respect to a packing lattice. Unfortunately, we have no idea about the right dimension of maximal "free" planes. It seems to be an interesting problem to narrow the gap between the bounds given in the theorems.

In non-lattice ball packings the situation is quite different. Recently it was shown in [8] that for any fixed d and $\epsilon > 0$, there exists a periodic packing set $X(d,\epsilon)$ of B^d such that the length of any segment contained in $\mathbb{R}^d \setminus (X(d,\epsilon) + \epsilon B^d)$ is bounded from above by a constant $c(d,\epsilon)$. So "free" planes are lattice phenomena.

We want to remark that in [14] the covering minima have been introduced in a more general setting, namely with respect to arbitrary norms. For more information on covering minima we refer to [14] and [16]. For a general introduction to lattices, homogeneous and inhomogeneous minima as well as to ball packings we refer to [6] and [19].

2. Proofs of the Theorems

The main ingredient of the proof of Theorem 1.1 is a Korkin-Zolotarev reduced basis of a lattice $\Lambda \subset \mathbb{R}^d$ which can be recursively defined as follows (cf. e.g. [16], [5]):

Definition 2.1. *A basis $\mathbf{a}_1, \dots, \mathbf{a}_d$ of a lattice is Korkin-Zolotarev reduced if*

1. $\|\mathbf{a}_1\| = \lambda_1(\Lambda)$,
2. $\gamma_i = \frac{\langle \mathbf{a}_i, \mathbf{a}_1 \rangle}{\|\mathbf{a}_1\|^2}$ *satisfies* $|\gamma_i| \le \frac{1}{2}$, $i = 2, \dots, d$,
3. *the projections* $\mathbf{a}_i | \mathrm{lin}(\mathbf{a}_1)^\perp = \mathbf{a}_i - \gamma_i \mathbf{a}_1$, $i = 2, \dots, d$, *yield a Korkin-Zolotarev reduced basis of the lattice* $\Lambda^{d-1} = \Lambda | \mathrm{lin}(\mathbf{a}_1)^\perp$.

The idea behind such a Korkin-Zolotarev reduced basis is to find a basis which is "as short and orthogonal as possible". It was shown in [15] that

Theorem 2.1 (Korkin-Zolotarev)**.** *Every lattice $\Lambda \subset \mathbb{R}^d$ has a Korkin-Zolotarev reduced basis $\mathbf{a}_1, \dots, \mathbf{a}_d$. Moreover, the homogeneous minima of the lattices $\Lambda^{d-i} = \Lambda | \mathrm{lin}\{\mathbf{a}_1, \dots, \mathbf{a}_i\}^\perp$, $i = 1, \dots, d-1$ and $\Lambda^d = \Lambda$ satisfy*

$$\lambda_1(\Lambda^{d-(i+1)}) \ge \frac{\sqrt{3}}{2} \lambda_1(\Lambda^{d-i})$$

and

$$\lambda_1(\Lambda^{d-(i+2)}) \ge \sqrt{\frac{2}{3}} \lambda_1(\Lambda^{d-i}).$$

Now we are ready for the proof of the first theorem.

Proof of Theorem 1.1. Let $\Lambda \subset \mathbb{R}^d$ be a lattice with a Korkin-Zolotarev reduced basis $\mathbf{a}_1, \ldots, \mathbf{a}_d$ and let Λ^k, $k = 1, \ldots, d$, be the corresponding lattices defined in Theorem 2.1. By the definition of $\mu_i(\Lambda)$ and Theorem 2.1 we have

$$\begin{aligned} 2\frac{\mu_{d-k}(\Lambda)}{\lambda_1(\Lambda)} &\geq 2\frac{\mu(\Lambda^{d-k})}{\lambda_1(\Lambda^{d-k})} \cdot \frac{\lambda_1(\Lambda^{d-k})}{\lambda_1(\Lambda)} \\ &\geq 2\frac{\mu(\Lambda^{d-k})}{\lambda_1(\Lambda^{d-k})} \cdot \sqrt{\frac{2}{3}}^{\lfloor \frac{k}{2} \rfloor} \left(\frac{\sqrt{3}}{2}\right)^{k \bmod 2}. \end{aligned} \tag{2.1}$$

Next, depending on the index k and the dimension d listed in Theorem 1.1, we apply to the quotient $2\frac{\mu(\Lambda^{d-k})}{\lambda_1(\Lambda^{d-k})}$ various well-known lower bounds on the ratio of covering radius to the packing radius of a lattice.

For any 3-dimensional lattice $\tilde{\Lambda}$ it was shown in [2] that

$$\frac{2\mu(\tilde{\Lambda})}{\lambda_1(\tilde{\Lambda})} \geq \sqrt{\frac{5}{3}}. \tag{2.2}$$

Thus, by (2.1) we have for $d = 5$ and $k = 2$

$$\frac{2\mu_{5-2}(\Lambda)}{\lambda_1(\Lambda)} \geq 2\frac{\mu(\Lambda^3)}{\lambda_1(\Lambda^3)} \cdot \sqrt{\frac{2}{3}} \geq \sqrt{\frac{10}{9}} > 1.$$

In other words,

$$\mu_{5-2}(\Lambda) > \frac{\lambda_1(\Lambda)}{2} \tag{2.3}$$

for every 5-dimensional lattice. For higher dimensions we use a result of [18] on the distance between the center and a vertex of a Voronoï-cell which, in our terminology, reads

$$\frac{2\mu(\tilde{\Lambda})}{\lambda_1(\tilde{\Lambda})} \geq \sqrt{\frac{2d}{d+1}} \tag{2.4}$$

for every d-dimensional lattice $\tilde{\Lambda}$. Hence for $k = 2$ we get

$$2\frac{\mu(\Lambda^{d-2})}{\lambda_1(\Lambda^{d-2})} \geq \sqrt{\frac{2d-4}{d-1}}$$

and therefore by (2.1), for $d \geq 6$,

$$\mu_{d-2}(\Lambda) > \frac{\lambda_1(\Lambda)}{2}.$$

Together with (2.3) we have proven the case i) of the theorem.

For the second case ($d = 8$, $k = 3$) we use a result of [12], saying

$$2\frac{\mu(\tilde{\Lambda})}{\lambda_1(\tilde{\Lambda})} \geq \sqrt{\frac{3}{2} + \frac{\sqrt{13}}{6}} \tag{2.5}$$

for every 5-dimensional lattice $\tilde{\Lambda}$.

Finally, for d sufficiently large and $k = 4$ we apply an inequality of [13]

$$2\frac{\mu(\tilde{\Lambda})}{\lambda_1(\tilde{\Lambda})} > 1.51, \quad \dim(\tilde{\Lambda}) \text{ large}, \tag{2.6}$$

to the quotient $2\frac{\mu(\Lambda^{d-k})}{\lambda_1(\Lambda^{d-k})}$ in (2.1).

All the inequalities (2.2), (2.4), (2.5), and (2.6) can be found with proofs in [19]. □

The proof also gives a description of the k-dimensional "free" planes. Namely, with the notation of Theorem 2.1, let $L^k = \mathrm{lin}\{\mathbf{a}_1, \dots, \mathbf{a}_k\}$ and let $\mathbf{p}_k \in \mathrm{lin}\{\mathbf{a}_1, \dots, \mathbf{a}_k\}^\perp$ be a point with

$$\mathrm{dist}(\mathbf{p}_k, \Lambda^{d-k}) = \mu(\Lambda^{d-k}),$$

where $\mathrm{dist}(X, Y)$ indicates the distance between X and Y. Then we have

$$\mathrm{dist}(\mathbf{p}_k + L^k, \Lambda) = \mu(\Lambda^{d-k}) > \frac{\lambda_1(\Lambda)}{2}$$

for the considered cases of d and k (cf. (2.1)).

Let us mention that by a result of [3] it is known that there exist lattices $\tilde{\Lambda} \subset \mathbb{R}^d$ with

$$2\frac{\mu(\tilde{\Lambda})}{\lambda_1(\tilde{\Lambda})} \leq 2 + o(1).$$

Thus with the method used in the proof, Theorem 1.1 can not be much improved.

To prove Theorem 1.2 we just combine two beautiful results from the theory of transference theorems.

Proof of Theorem 1.2. Let $\Lambda \subset \mathbb{R}^d$ be a d-dimensional lattice and let Λ^* be its dual,

$$\Lambda^* = \{\mathbf{v} \in \mathbb{R}^d : \langle \mathbf{u}, \mathbf{v} \rangle \in \mathbb{Z}, \text{ for all } \mathbf{u} \in \Lambda\}.$$

It was shown by Conway and Thompson (see page 46 of [17]) that there exists a constant β_1 such that for every dimension d one can find a lattice $\tilde{\Lambda} \subset \mathbb{R}^d$ with

$$\lambda_1(\tilde{\Lambda}) \cdot \lambda_1(\tilde{\Lambda}^*) \geq \beta_1 d.$$

By definition $[\lambda_1(\tilde{\Lambda}^*)]^{-1}$ is the maximal distance between parallel $(d-1)$-dimensional lattice hyperplanes of $\tilde{\Lambda}$ and thus (see [14])

$$2\mu_1(\tilde{\Lambda}) = 1/\lambda_1(\tilde{\Lambda}^*).$$

Therefore we have

$$2\frac{\mu_1(\tilde{\Lambda})}{\lambda_1(\tilde{\Lambda})} = \frac{1}{\lambda_1(\tilde{\Lambda}) \cdot \lambda_1(\tilde{\Lambda}^*)} \leq \frac{1}{\beta_1 d}. \tag{2.7}$$

On the other hand, [1] proved that there exists a constant $\beta_2 > 0$ such that

$$\mu(\Lambda) \cdot \lambda_1(\Lambda^*) \leq \beta_2 d$$

for every lattice $\Lambda \subset \mathbb{R}^d$. Or with the first covering minima we may write

$$\mu(\Lambda) \leq 2\beta_2\, d \cdot \mu_1(\Lambda). \tag{2.8}$$

Now for a given lattice $\Lambda \subset \mathbb{R}^d$ and an index $k \in \{1, \dots, d\}$ let L^{d-k} be a $(d-k)$-dimensional lattice hyperplane such that for the lattice $\Lambda^k = \Lambda|(L^{d-k})^\perp$ holds

$$\mu_k(\Lambda) = \mu(\Lambda^k).$$

Then we have

$$\mu_1(\Lambda^k) \leq \mu_1(\Lambda)$$

and, applying (2.8) to the lattice Λ^k,

$$\mu_k(\Lambda) \leq 2\beta_2\, k \cdot \mu_1(\Lambda), \quad 1 \leq k \leq d.$$

Hence for a lattice $\tilde{\Lambda}$ satisfying (2.7) we may deduce

$$2\frac{\mu_k(\tilde{\Lambda})}{\lambda_1(\tilde{\Lambda})} \leq \frac{4\beta_2 k \cdot \mu_1(\tilde{\Lambda})}{\lambda_1(\tilde{\Lambda})} \leq \frac{2\beta_2 k}{\beta_1 d}.$$

In other words, for $k < \frac{\beta_1}{2\beta_2}d$ the lattice $\tilde{\Lambda}$ does not contain a "free" $(d-k)$-dimensional plane. □

3. Some examples

Here we study "free" planes for some classical d-dimensional lattice ball packings. To this end let $\mathbf{q}_d = (\frac{1}{2}, \ldots, \frac{1}{2}) \in \mathbb{R}^d$,

$$D^d = \{\mathbf{z} \in \mathbb{Z}^d : \ z_1 + \cdots + z_d = 0 \bmod 2\},$$

$$A^d = \{\mathbf{z} \in \mathbb{Z}^{d+1} : \ z_1 + \cdots + z_{d+1} = 0\},$$

$$E^8 = \{\mathbf{x} \in \mathbb{Z}^8 \cup (\mathbf{q}_8 + \mathbb{Z}^8) : \ x_1 + \cdots + x_8 = 0 \bmod 2\},$$

$$E^7 = \{\mathbf{x} \in E^8 : \ x_1 + \cdots + x_8 = 0\},$$

and

$$E^6 = \{\mathbf{x} \in E^8 : \ x_1 = x_2 = x_3\}.$$

The lattices A^3, D^4, D^5, E^6, E^7, and E^8 are the unique lattices producing the densest lattice ball packing in the corresponding dimensions. For the homogeneous minima we have (cf. [4] by inspection):

$$\frac{\lambda_1(D^d)}{2} = \frac{\lambda_1(A^d)}{2} = \frac{\lambda_1(E^6)}{2} = \frac{\lambda_1(E^7)}{2} = \frac{\lambda_1(E^8)}{2} = \frac{1}{\sqrt{2}}. \tag{3.1}$$

Let $H^{i,d}$ be the i-dimensional plane in $\mathbb{R}^d$ given by

$$H^{i,d} = \left\{ \left(\tfrac{1}{2}, \ldots, \tfrac{1}{2}, x_{d-i+1}, \ldots, x_d\right) : \ x_j \in \mathbb{R} \right\}.$$

Then we have

$$\operatorname{dist}(H^{i,d}, \mathbb{Z}^d) = \frac{\sqrt{d-i}}{2}$$

and therefore, for $k \geq 2$,

$$\begin{aligned} \mu_k(D^d) \ &\geq \ \operatorname{dist}(H^{d-k,d}, D^d) \geq \operatorname{dist}(H^{d-k,d}, \mathbb{Z}^d) \\ &= \frac{\sqrt{k}}{2} \geq \frac{\lambda_1(D^d)}{2}. \end{aligned}$$

This means that we can always find a $(d-2)$-dimensional plane intersecting none of the balls in the interior of the lattice ball packing associated to D^d.

Next, with $H = \{\mathbf{x} \in \mathbb{R}^{d+1} : x_1 + \cdots + x_{d+1} = 0\}$ we find analogously for the lattice A^d and $k \geq 2$

$$\mu_k(A^d) \geq \operatorname{dist}(H^{d+1-k,d+1} \cap H, A^d)$$
$$\geq \operatorname{dist}(H^{d+1-k,d+1}, \mathbb{Z}^{d+1}) \geq \frac{\lambda_1(A^d)}{2}.$$

For the lattice E^8 we set

$$H^4 = \left\{ \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, x_5, \ldots, x_8\right) : x_j \in \mathbb{R} \right\}$$

and get

$$\mu_4(E^8) \geq \operatorname{dist}(H^4, E^8) = \frac{1}{\sqrt{2}} = \frac{\lambda_1(E^8)}{2}.$$

By intersecting H^4 with the hyperplane $\{\mathbf{x} \in \mathbb{R}^8 : x_1 + \cdots + x_8 = 0\}$ we also get

$$\mu_3(E^7) \geq \frac{\lambda_1(E^7)}{2}.$$

Finally, for the lattice E^6 let

$$H^3 = \left\{ \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, 0, x_6, x_7, x_8\right) : x_j \in \mathbb{R} \right\}.$$

Then $H^3 \subset \operatorname{lin}(E^6)$ and it is easy to see that

$$\mu_3(E^6) \geq \operatorname{dist}(H^3, E^6) = \frac{1}{\sqrt{2}} = \frac{\lambda_1(E^6)}{2}.$$

In particular these calculations imply

Remark 3.1. *The densest lattice packings of B^d, $d = 3, \ldots, 8$, contain a $\lfloor \frac{d}{2} \rfloor$-dimensional lattice plane L such that for a suitable vector $\mathbf{t}$ the affine plane $\mathbf{t} + L$ does not meet any of the balls of the packing in their interior.*

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