On orbit configuration spaces of spheres

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Abstract

The orbit configuration space $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ is the space of all ordered n-tuples of points on the k-sphere such that no two of them are identical or antipodal. The cohomology algebra of $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$, with integer coefficients, is here determined completely, and described in terms of generators, bases and relations. To this end, we analyze the cohomology spectral sequence of a fibration $\mathrm{F}_{\mathbb{Z}_2}(S^k, n) \longrightarrow S^k$, where the fiber — in contrast to the situation for the classical configuration space $\mathrm{F}(S^k, n)$ — is not the complement of a linear subspace arrangement. Analogies to the arrangement case, however, are crucial for getting a complete description of its cohomology.

Key words: orbit configuration spaces, integral cohomology algebra, subspace arrangements, Leray-Serre spectral sequence

1 Introduction

The ordered *configuration spaces*

$$\mathrm{F}(M, n) \quad := \quad \{(x_1, x_2, \ldots, x_n) \in M^n : x_i \neq x_j \text{ for all } i \neq j\}$$

of a manifold M were first studied by Fadell & Neuwirth [FN] in 1962. The cohomology of such spaces has received a lot of attention, at least since the work by Arnol'd [A] related to Hilbert's 13th problem. The homology of the

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loop spaces over configuration spaces has been studied intensively in the last ten years due to the Lie algebra structures it carries; see [C3,CG] for surveys.

Recently, orbit configuration spaces and their cohomology have attracted interest, e. g. because of an analogy to fiber-type arrangements [C] and as a tool to construct equivariant loop spaces [X]. Let G be a finite group acting freely on a manifold M. Then the *orbit configuration space* $\mathrm{F}_G(M,n)$ is the space of all n-tuples of points in M whose orbits are pairwise disjoint:

$$\mathrm{F}_G(M,n) \quad := \quad \{(x_1, x_2, \ldots, x_n) \in M^n : Gx_i \neq Gx_j \text{ for all } i \neq j\}.$$

Equivalently, $\mathrm{F}_G(M,n)$ is the $|G|^n$-fold covering of the ordered configuration space $\mathrm{F}(M/G,n)$ of M/G induced by the covering map $M \longrightarrow M/G$.

The following is a very brief summary of the explicit results on the cohomology for some basic examples both of classical and of orbit configuration spaces. This is meant to explain why the orbit configuration spaces of the k-spheres now provide the logical next step to look at, why one is interested in their cohomology algebras with integer coefficients, and how this line of research is intimately related to the study of subspace arrangements.

$\mathrm{F}(\mathbb{R}^2,n) \cong \mathrm{F}(\mathbb{C},n)$ is the complement of a linear arrangement of hyperplanes ("the braid arrangement") in $\mathbb{C}^n$. Its cohomology algebra has no torsion; it is generated by elements associated with individual hyperplanes, and its ideal of relations is generated by three-term relations that correspond to triples of subspaces with the same pairwise intersection [A].

Arnol'd's investigations started intensive work on complex hyperplane arrangements, which led to Deligne's theorem [D] on $K(\pi,1)$-complements, and to Orlik & Solomon's [OS] combinatorial description of the cohomology algebras (with integer coefficients) of general complex hyperplane arrangements (see also [BZ,FZ1]).

$\mathrm{F}(\mathbb{R}^k,n)$ is the complement of a linear arrangement of subspaces of codimension k in $\mathbb{R}^{nk}$. The cohomology of these spaces was obtained by Cohen [C1,C2]. It is again torsion free, with generators of degree $k-1$ corresponding to individual subspaces, and relations corresponding to triples with the same pairwise intersection.

The spaces $\mathrm{F}(\mathbb{R}^k,n)$ were motivating examples for the "k-arrangements" introduced by Goresky & MacPherson [GM, Part III]; this in turn was a starting point for an intensive study of general subspace arrangements, as surveyed by Björner in [Bj2]. The cohomology algebra of $\mathrm{F}(\mathbb{R}^k,n)$ can now be recovered from general results on the cohomology of arrangement complements [BZ] [FZ1] [dLS].

In [Z] and [MS] it was noted that important information is hidden in the *multiplicative structure* of the cohomology of arrangements, as opposed to just in their cohomology as graded groups. See also Deligne, Goresky & MacPherson [DGM] and de Longueville & Schultz [dLS] for very recent work.

$\mathrm{F}(S^k, n)$ was already studied by Fadell & Neuwirth [FN] in the early sixties. The fibrations

$$\Pi : \quad \begin{array}{rcl} \mathrm{F}(S^k, n) & \longrightarrow & S^k \\ (x_1, \ldots, x_n) & \longmapsto & x_n \, . \end{array}$$

appear in their work. They are essential for deriving descriptions of homotopy groups of configuration spaces of spheres. Cohen & Taylor [CT1,CT2] determined the cohomology algebras of $\mathrm{F}(S^k, n)$ with coefficients in a field of characteristic unequal to 2, while the explicit result with integer coefficients – with 2-torsion appearing in the case when k is even – was apparently first published in [FZ2]. Thus it appears that there is extra information to be derived from the cohomology with *integer coefficients*.

$\mathrm{F}_{\mathbb{Z}_2}(\mathbb{R}^k \backslash \{0\}, n)$, with the defining $\mathbb{Z}_2$-action given by scalar multiplication with -1 on $\mathbb{R}^k \backslash \{0\}$, is probably the easiest example of an orbit configuration space. It is the complement of an arrangement of linear subspaces (compare the arrangement $\mathcal{B}_n^{(k)}$ described after Remark 2). A complete description of its cohomology algebra again follows from the general theory of arrangement complements.

The case of $k = 2$ is also the special case $r = 2$ of the orbit configuration spaces $\mathrm{F}_{\mathbb{Z}_r}(\mathbb{R}^2 \backslash \{0\}, n)$, where the group of r-th roots of unity acts on $\mathbb{R}^2 \backslash \{0\} = \mathbb{C}^*$ by multiplication. These spaces were studied in recent work by D. Cohen [C].

$\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$, with the $\mathbb{Z}_2$-action given by the antipodal map on S^k, presents the natural next step for the investigation of orbit configuration spaces and their cohomology. Explicitly, it is given by

$$\mathrm{F}_{\mathbb{Z}_2}(S^k, n) \quad := \quad \{(x_1, x_2, \ldots, x_n) \in (S^k)^n : x_i \neq \pm x_j \text{ for all } i \neq j\}.$$

Equivalently, this is the space of all n-tuples of points on S^k whose orbits under the antipodal map $\nu : S^k \longrightarrow S^k$, $x \longmapsto -x$ are disjoint. Also equivalently, $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ is the 2^n-fold covering space of the ordered configuration space $\mathrm{F}(\mathbb{R}\mathrm{P}^k, n)$ of $\mathbb{R}\mathrm{P}^k$ that is induced by the covering map $S^k \longrightarrow \mathbb{R}\mathrm{P}^k$.

In the following, we determine explicitly the cohomology algebras

$$H^*(\mathrm{F}_{\mathbb{Z}_2}(S^k, n); \mathbb{Z})$$

with integer coefficients. The results are given in three ways:

- We give presentations in terms of generators and relations. See Theorem 14 for the case of odd $k \geq 3$, Theorem 17 for the case of even $k \geq 2$.

- The cohomology algebras are $\mathbb{Z}$-algebras of finite ranks, which can also be described in terms of bases in the case where k is odd and no torsion occurs (the relevant data are given in Proposition 8(2)), resp. in terms of minimal sets of generators in the case where k is even (cf. Proposition 16).

- The results turn out to be closely related to the cohomology of certain (linear) subspace arrangements of type B. Thus in a certain sense, the spaces $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ are "type B configuration spaces of the k-sphere." The relation is discussed at the end of Section 4.

Our methods in the following are rather elementary, extending those of [FZ2]. They are based on a "Fadell-Neuwirth type" fibration, which has a k-sphere as its base space (Section 2). The fiber $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \backslash \{0\}, n-1)$ is an arrangement of three types of codimension k subspaces, one type however being non-linear. Its cohomology algebra is the topic of Section 3. The essential new step/ingredient is to define cohomology generators for the non-linear subspaces as images of generators associated with the linear subspaces in the arrangement. We further deal with the non-linearity by setting up maps to linear arrangements of three subspaces that transport cohomology generators and induce three-term relations; compare the proofs of Lemma 7 and of Proposition 11. In Section 4 we give an analysis of the Leray-Serre spectral sequence associated to the considered fibre map, which leads to a complete description of the cohomology algebra of $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ in generators and relations (Theorems 14 and 17).

2 A fibre map on $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$

The orbit configuration space $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ is the total space of a locally trivial fibre bundle [X, Thm. 2.2.2] given by the projection

$$\begin{array}{rrcl} \Pi : & \mathrm{F}_{\mathbb{Z}_2}(S^k, n) & \longrightarrow & S^k \\ & (x_1, \ldots, x_n) & \longmapsto & x_n \,. \end{array}$$

Our main objective is to investigate the Leray-Serre spectral sequence $E_*(\Pi)$ associated to this fibre map. It converges to the cohomology of the total space $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$. The E_2-term, $E_2(\Pi)$, takes a particularly simple form since the base space of the fibre bundle is simply connected for $k \geq 2$ and its cohomology is torsion free: Thus we need only the cohomology of base space and fibre of Π as the input data for the spectral sequence.

The fibre over a point x_n of S^k is given by configurations of pairwise distinct and non-antipodal points on S^k that avoid the orbit $\{\pm x_n\}$:

$$\begin{aligned}\Pi^{-1}(\{x_n\}) = \{\, (x_1,\ldots,x_{n-1}) \in (S^k)^{n-1} \mid {} & x_i \neq \pm x_j \text{ for } i \neq j, \\ & x_i \neq \pm x_n \text{ for } 1 \leq i \leq n-1 \} \,.\end{aligned}$$

Configurations on S^k that avoid a pair of antipodal points can be viewed as configurations on $\mathbb{R}^k$ that avoid $\{0\}$. Via stereographic projection, the antipodal action of $\mathbb{Z}_2$ on S^k translates into a $\mathbb{Z}_2$-action generated by the following involution φ on $\mathbb{R}^k\backslash\{0\}$:

$$\varphi : x \longmapsto -\frac{1}{\|x\|^2}\, x \,.$$

Hence the fibre of Π is homeomorphic to the orbit configuration space

$$\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n-1) \,.$$

Unlike in the case of ordered configuration spaces of spheres, the fibre is *not* the complement of an arrangement of linear subspaces: Besides the linear subspaces

$$\begin{aligned} U^+_{i,j} &:= \{\, (x_1,\ldots,x_{n-1}) \in (\mathbb{R}^k)^{n-1} \mid x_i = x_j \,\}\,, \qquad 1 \leq i < j \leq n-1\,, \\ U_i &:= \{\, (x_1,\ldots,x_{n-1}) \in (\mathbb{R}^k)^{n-1} \mid x_i = 0 \,\}\,, \qquad 1 \leq i \leq n-1\,, \end{aligned}$$

the nonlinear surfaces

$$U^-_{i,j} := \{\, (x_1,\ldots,x_{n-1}) \in (\mathbb{R}^k)^{n-1} \mid x_i = \varphi(x_j) \,\}\,, \quad 1 \leq i < j \leq n-1\,,$$

have to be excluded from the ambient space $(\mathbb{R}^k)^{n-1}$ in order to obtain the fibre $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n-1)$.

3 The configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n)$

In this section we determine for all $k \geq 2$ and $n \geq 1$ the cohomology of the orbit configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n)$, that is, of the fibre of the projection Π.

Proposition 1 *For $k > 2$, $n \geq 1$, the integer cohomology of the orbit configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n)$ can as a graded group be described as*

$$H^*(\,\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n)\,,\,\mathbb{Z}\,) \;\cong\; \bigotimes_{j=1}^{n} (\,\mathbb{Z}(0) \,\oplus\, \mathbb{Z}(k{-}1)^{\,2j-1}\,)\,,$$

where $\mathbb{Z}(t)$ denotes an infinite cyclic summand in degree t.

PROOF. We establish the isomorphism by induction on n, along with the fact that $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n)$ is simply connected for $k > 2$.

The assertions are true for $n = 1$: $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, 1) \cong \mathbb{R}^k\backslash\{0\} \simeq S^{k-1}$.
For $n > 1$, define $\Pi_{[n-1]} : \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n) \longrightarrow \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n{-}1)$ by projecting every point configuration to its first $n{-}1$ points. With $Q^{n-1}_{\langle\varphi\rangle}$ denoting a point set that contains 0 and $n{-}1$ pairwise distinct orbits of φ in $\mathbb{R}^k\backslash\{0\}$, we obtain a locally trivial fibre bundle [X, Thm. 2.2.2] of the form

$$\begin{array}{ccc}
\bigvee\limits_{2(n-1)+1} S^{k-1} \;\simeq\; \mathbb{R}^k\backslash Q^{n-1}_{\langle\varphi\rangle} & \longrightarrow & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n) \\
& & \big\downarrow{\scriptstyle \Pi_{[n-1]}} \\
& & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n{-}1)\,.
\end{array}$$

We analyze the associated Leray-Serre spectral sequence $E_*(\Pi_{[n-1]})$: By induction hypothesis, the base space of the fibre bundle is simply connected and its cohomology is torsion free. This yields the following description of the E_2-term:

$$\begin{aligned}
E_2^{*,*} \cong \quad & H^*(\bigvee_{2(n-1)+1} S^{k-1}) \quad\otimes\quad H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n{-}1)) \\
\cong & (\,\mathbb{Z}(0) \,\oplus\, \mathbb{Z}(k{-}1)^{2n-1}\,) \quad\otimes\quad \bigotimes_{j=1}^{n-1} (\,\mathbb{Z}(0) \,\oplus\, Z(k{-}1)^{\,2j-1}\,)\,.
\end{aligned}$$

The placement of non-zero tableau entries shows that there is no non-trivial differential on E_2 or on any higher sequence term. Hence the spectral sequence collapses in its second term, and a straightforward reconstruction argument yields the result on the cohomology groups of $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n)$.

Moreover, base space and fibre of the considered fibre bundle are simply connected. We refer to standard arguments involving the long exact sequence of homotopy groups of a fibre bundle and conclude that the total space is as well simply connected. $\square$

Remark 2 We will establish in Remark 10 that a similar result holds for the cohomology of $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^2\backslash\{0\},n)$. These configuration spaces are not simply connected and somewhat more involved arguments have to be used in order to assure that the coefficient systems induced by the action of the fundamental group of the base space on the cohomology of the fibre are simple.

Let $\mathcal{B}_n^{(k)}$ denote the arrangement of linear subspaces in $\mathbb{R}^{k\cdot n}$ formed by

$$\begin{aligned} V_{i,j}^+ &:= \{\,(x_1,\ldots,x_n)\in(\mathbb{R}^k)^n \,|\, x_i = x_j\,\},\ 1\le i<j\le n\,,\\ V_{i,j}^- &:= \{\,(x_1,\ldots,x_n)\in(\mathbb{R}^k)^n \,|\, x_i = -x_j\,\},\ 1\le i<j\le n\,,\\ V_i\ &:= \{\,(x_1,\ldots,x_n)\in(\mathbb{R}^k)^n \,|\, x_i = 0\,\},\ 1\le i\le n\,. \end{aligned}$$

This arrangement is a codimension k version of the arrangement of reflecting hyperplanes $\mathcal{B}_n$ associated with the Coxeter group of type B_n (cf. [OT, Def. 6.24]). We denote its complement $\mathbb{R}^{k\cdot n}\backslash\bigcup\mathcal{B}_n^{(k)}$ by $\mathcal{M}(\mathcal{B}_n^{(k)})$. Complete information on the integer cohomology algebra of $\mathcal{M}(\mathcal{B}_n^{(k)})$ can be obtained from well-established theory on the cohomology of subspace arrangements [Bj2,BZ,FZ1].

We restrict our attention for the moment to the cohomology *groups* of $\mathcal{M}(\mathcal{B}_n^{(k)})$. A full description can be derived by adapting word by word the arguments given in the proof of Proposition 1: Projection of $\mathcal{M}(\mathcal{B}_n^{(k)})$ onto $\mathcal{M}(\mathcal{B}_{n-1}^{(k)})$ by "forgetting" the last k coordinates of a point yields a locally trivial fibre bundle whose fibre is homeomorphic to $\mathbb{R}^k$ with $2n-1$ points removed.

We can state the following observation:

Proposition 3 *For $k>2$ and $n\ge 1$, the integer cohomology of the orbit configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$ is isomorphic as a graded group to the integer cohomology of the arrangement $\mathcal{B}_n^{(k)}$:*

$$H^*(\,\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n),\,\mathbb{Z}\,)\ \cong\ H^*(\,\mathcal{M}(\mathcal{B}_n^{(k)}),\,\mathbb{Z}\,)\,.$$

Remark 4 Again, a similar result holds for $k=2$. For a discussion of the cohomology groups of $\mathcal{M}(\mathcal{B}_n^{(2)})$ in the broader context of fibre-type arrangements see [FR].

Consider the inclusions of $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$ into subspace complements:

$$\begin{aligned} \phi_i\ &:\quad \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)\hookrightarrow\mathcal{M}(\{U_i\})\,, \qquad 1\le i\le n\,,\\ \phi_{i,j}\ &:\quad \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)\hookrightarrow\mathcal{M}(\{U_{i,j}^+\})\,, \qquad 1\le i<j\le n\,. \end{aligned}$$

Here and in the following, we adopt notation from the arrangement context and denote by $\mathcal{M}(\,\cdot\,)$ the complement of the listed spaces in their natural ambient space.

Moreover, define involutions A_i on $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$, $1\leq i\leq n$, by applying φ to the i-th point in a configuration:

$$\begin{aligned} A_i\,:\quad \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) &\longrightarrow \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) \\ (x_1,\ldots,x_i,\ldots,x_n) &\longmapsto (x_1,\ldots,\varphi(x_i),\ldots,x_n)\,. \end{aligned}$$

Let $\hat{c}_i$ be a canonical generator of $H^{k-1}(\mathcal{M}(\{U_i\}))$, $1\leq i\leq n$, and $\hat{c}_{i,j}$ a canonical generator of $H^{k-1}(\mathcal{M}(\{U^+_{i,j}\}))$. Compare [BZ, Sect. 9] to see that we can actually speak of canonical cohomology generators in this context, assuming that we fixed a "frame of hyperplanes" for the subspaces U_i and $U^+_{i,j}$ respectively, in the obvious way.

We define a set of cohomology classes on $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$ by restricting cohomology classes from the subspace complements and by applying maps induced by the involutions A_i, respectively:

Definition 5 *With the notations introduced above, we define cohomology classes on* $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$ *by:*

$$\begin{aligned} c_i &:= \phi_i^*(\hat{c}_i)\,, \quad 1\leq i\leq n\,, \\ c^+_{i,j} &:= \phi^*_{i,j}(\hat{c}_{i,j})\,,\ 1\leq i<j\leq n\,, \\ c^-_{i,j} &:= A_i^*(c^+_{i,j})\,,\ 1\leq i<j\leq n\,. \end{aligned}$$

It turns out that this definition of the cohomology classes c_i, $c^+_{i,j}$ and $c^-_{i,j}$ is compatible with projections of configuration spaces:

Lemma 6 *Let the map* $\Pi_S : \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) \longrightarrow \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},|S|)$, $S\subseteq[n]$, *be given by projecting a point configuration to the coordinates with index in* S. *Then*

$$\begin{aligned} \Pi_S^*(c_i) &= c_i\,, \\ \Pi_S^*(c^+_{i,j}) &= c^+_{i,j}\,, \\ \Pi_S^*(c^-_{i,j}) &= c^-_{i,j} \qquad \text{for } i,j\in S,\ \ i\neq j\,, \end{aligned}$$

where, without further notice, the cohomology classes belong to the cohomology of the respective configuration spaces.

PROOF. The assertions follow from the corresponding results for subspace complements, and from the fact that the projections commute with the involutions A_i for $i \in S$. $\square$

Lemma 7 *For the cohomology classes of* Definition 5*, the following identities hold:*

$$
\begin{array}{lll}
(i) & A_j^*(c_i) = c_i & \textit{for } j \neq i\,,\ \ 1 \leq i,j \leq n\,,\\
(ii) & A_r^*(c_{i,j}^+) = c_{i,j}^+ & \textit{for } r \neq i,j\,,\ \ 1 \leq r,i,j \leq n\,,\\
(iii) & A_i^*(c_i) = (-1)^k\, c_i & 1 \leq i \leq n\,,\\
(iv) & A_j^*(c_{i,j}^+) = (-1)^k c_{i,j}^- & 1 \leq i,j \leq n\,.
\end{array}
$$

PROOF. (i) For $j \neq i$, $\Pi_i \circ A_j = \Pi_i \,:\, \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) \longrightarrow \mathbb{R}^k\backslash\{0\}$, where Π_i denotes the projection defined in Lemma 6. With the compatibility results proven there, we obtain: $A_j^*(c_i) \;=\; A_j^*\,\Pi_i^*(c_i) \;=\; \Pi_i^*(c_i) \;=\; c_i$, where again c_i is to be understood as a cohomology class for the configuration spaces of n points, and of 1 point, respectively.

(ii) For $r \neq i,j$, $\Pi_{i,j}\circ A_r = \Pi_{i,j} \,:\, \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) \longrightarrow F_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},2)$, and the statement follows as above.

(iii) Let $\sigma_i \,:\, \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) \longrightarrow S^{k-1}$ be defined by projection of a point configuration to its i-th coordinate and further to the unit sphere in $\mathbb{R}^k$. Then the involution A_i restricts to the antipodal map A on S^{k-1}: $\sigma_i \circ A_i \;=\; A \circ \sigma_i$. Thus, again using Lemma 6, we conclude that $A_i^*(c_i) \;=\; (-1)^k c_i$.

(iv) With Lemma 6 it is enough to establish the identity for the corresponding classes in the 2-point configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},2)$.

Step 1: $(A_2 \circ A_1)^*(c_{1,2}^+) \;=\; (-1)^k c_{1,2}^+$.
Consider the inclusion of $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},2)$ into the arrangement complement $\mathcal{M}(\{U_1,U_2,U_{1,2}^+\})$:

$$\phi \,:\, \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},2) \longrightarrow \mathcal{M}(\{U_1,U_2,U_{1,2}^+\})\,.$$

Obviously, the canonical cohomology generators $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_{1,2}$ corresponding to the subspaces $U_1,U_2,U_{1,2}^+$ map to c_1, c_2, $c_{1,2}^+$ under ϕ^*. It suffices to show that

$$(A_2 \circ A_1)^*(\tilde{c}_{1,2}) \;=\; (-1)^k \tilde{c}_{1,2}\,,$$

where A_i now denote the involutions extended to the arrangement complements. Consider the orthogonal projection of $\mathcal{M}(\{U_1,U_2,U_{1,2}^+\})$ along $U_{1,2}^+$ onto

${}^{\perp}U^+_{1,2}\backslash\{0\}$, combined with the retraction on the central sphere S of radius $\sqrt{2}$ in ${}^{\perp}U^+_{1,2}$, $p: \mathcal{M}(\{U_1, U_2, U^+_{1,2}\}) \longrightarrow S$. On S, $\|(x,y)\| = \sqrt{2}$, and $x + y = 0$. Hence,

$$2 \;=\; \sum_{i=1}^{k} x_i^2 \;+\; \sum_{i=1}^{k} y_i^2 \;=\; 2\,\sum_{i=1}^{k} x_i^2\,,$$

and we conclude that $\|x\| = \|y\| = 1$. Thus $A_2 \circ A_1$ restricts to the antipodal map on S:

$$(A_2 \circ A_1)\,(x,y) \;=\; (-\,\frac{1}{\|x\|^2}\;x\,,-\,\frac{1}{\|y\|^2}\;y\,) \;=\; (-x,-y) \quad \text{for } x,y \in S\,,$$

and we conclude that $(A_2 \circ A_1)^*(\tilde{c}_{1,2}) = (-1)^k \tilde{c}_{1,2}$.

Step 2: $A_2^*(c^+_{1,2}) \;=\; (-1)^k c^-_{1,2}$.
Since $A_2 \circ A_1 = A_1 \circ A_2$ on $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},2)$, the assertion of Step 1 reads:

$$A_2^* \circ A_1^*(c^+_{1,2}) \;=\; (-1)^k c^+_{1,2}\,.$$

Applying A_2 on both sides, and noting that by definition $A_1^*(c^+_{1,2}) = c^-_{1,2}$, yields

$$c^-_{1,2} \;=\; (-1)^k A_2^*(c^+_{1,2})\,. \qquad \square$$

Proposition 8
(1) *The cohomology algebra with integer coefficients of the orbit configuration space* $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$*,* $k > 2$*, is multiplicatively generated in degree* $k-1$ *by the cohomology classes*

$$c_i, \quad c^+_{i,j}, \quad c^-_{i,j},$$

for $1 \le i \le n$ *and* $1 \le i < j \le n$*, respectively.*
(2) *For ease of notation, set* $c^\epsilon_{0,j} := c_j$ *for* $\epsilon = \pm 1$*,* $1 \le j \le n$*. Then, a* $\mathbb{Z}$*-linear basis for the cohomology of* $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$*,* $k > 2$*, is given by cohomology classes*

$$c^{\epsilon_1}_{i_1,j_1} \;\cup\; \ldots \;\cup\; c^{\epsilon_t}_{i_t,j_t}\,,$$

where $1 \le t \le n$*,* $0 \le i_r < j_r \le n$*,* $\epsilon_r \in \{\pm 1\}$ *for* $r = 1,\ldots,t$*, and* $j_1 < \ldots < j_t$*.*

Remark 9 With part (2) of the preceding Proposition we establish a further analogy of the orbit configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$ to the subspace arrangement $\mathcal{B}^{(k)}_n$: Assuming a lexicographic order on the index “triples” (i,j,ϵ),

$0\leq i\leq n{-}1$, $1\leq j\leq n$, $\epsilon\in\{\pm 1\}$, we see that the linear basis is indexed with the elements of the broken circuit complex $\mathrm{BC}(\mathcal{B}_n^{(k)})$ associated with the arrangement $\mathcal{B}_n^{(k)}$ – the standard indexing set of a linear basis for the cohomology of this arrangement (compare [BZ, Thm. 7.2 & Sect. 9] and [FZ1, Thm. 5.2]).

PROOF. [Proof of Proposition 8] For both parts of the assertion, our proof will be by induction on n. Observe that the claims hold for $n{=}1$ with $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},1)=\mathbb{R}^k\backslash\{0\}$.

(1) As in the proof of Proposition 1 we consider the spectral sequence associated with the fibre bundle

$$\mathbb{R}^k\backslash Q^{n-1}_{\langle\varphi\rangle} \xrightarrow{\;j\;} \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) \xrightarrow{\;\Pi_{[n-1]}\;} \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n{-}1)\,.$$

Let us abbreviate the involved spaces by the classical notations for fibre bundles: E for the total space, B for the base space, and F for the fibre. The spectral sequence collapses in its second term and the two non-trivial rows of the corresponding sequence tableau read as follows:

$$\begin{aligned} E_2^{*,0} &\cong H^*(B)\,,\\ E_2^{*,k-1} &\cong H^{k-1}(F)\,\otimes\, H^*(B)\,. \end{aligned}$$

With $\Pi^*_{[n-1]}: H^*(B)\longrightarrow H^*(E)$ being injective [Bo, Thm. 14.2(a)], and

$$E_2^{*,k-1}\;\cong\; H^{*+k-1}(E)\,/\,\Pi^*_{[n-1]}(H^{*+k-1}(B))\,,$$

we see that $H^*(E)$ is multiplicatively generated by

- the images of multiplicative generators of $H^*(B)$ under $\Pi^*_{[n-1]}$,
- a choice of inverse images of generators of $H^{k-1}(F)$ under j^*.

By induction, $H^*(B)$ is multiplicatively generated by c_i, $c^+_{i,j}$ and $c^-_{i,j}$ for $1\leq i\leq n{-}1$, and $1\leq i<j\leq n{-}1$, respectively. Taking images under $\Pi^*_{[n-1]}$ yields the corresponding generators in $H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n))$ (cf. Lemma 6). It remains to show that the cohomology classes $j^*(c_n)$, $j^*(c^+_{i,n})$, $j^*(c^-_{i,n})$, for $1\leq i\leq n{-}1$, generate $H^{k-1}(F)$.

As described before, $Q^{n-1}_{\langle\varphi\rangle}=\{0,q_1,\varphi(q_1),\ldots,q_{n-1},\varphi(q_{n-1})\}$, where the points $q_1,\ldots,q_{n-1}$ are from distinct $\langle\varphi\rangle$-orbits of $\mathbb{R}^k\backslash\{0\}$. Consider the maps of fibre bundles

$$\begin{array}{ccccc}
\mathbb{R}^k\backslash\{0\} & \longrightarrow & \mathcal{M}(\{U_n\}) & \longrightarrow & (\mathbb{R}^k)^{n-1} \\
\uparrow{\scriptstyle \phi_n} & & \uparrow{\scriptstyle \phi_n} & & \uparrow \\
\mathbb{R}^k\backslash Q^{n-1}_{\langle\varphi\rangle} & \xrightarrow{j} & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) & \xrightarrow{\Pi_{[n-1]}} & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n-1)
\end{array}$$

$$\begin{array}{ccccc}
\mathbb{R}^k\backslash\{q_i\} & \longrightarrow & \mathcal{M}(\{U^+_{i,n}\}) & \longrightarrow & (\mathbb{R}^k)^{n-1} \\
\uparrow{\scriptstyle \phi_{i,n}} & & \uparrow{\scriptstyle \phi_{i,n}} & & \uparrow \\
\mathbb{R}^k\backslash Q^{n-1}_{\langle\varphi\rangle} & \xrightarrow{j} & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) & \xrightarrow{\Pi_{[n-1]}} & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n-1)
\end{array}$$

$$\begin{array}{ccccc}
\mathbb{R}^k\backslash Q^{n-1}_{\langle\varphi\rangle} & \xrightarrow{j} & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) & \xrightarrow{\Pi_{[n-1]}} & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n-1) \\
\uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle A_n} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathbb{R}^k\backslash Q^{n-1}_{\langle\varphi\rangle} & \xrightarrow{j} & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n) & \xrightarrow{\Pi_{[n-1]}} & \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n-1)
\end{array}$$

The first two diagrams show that $j^*(c_n)$ is the image of a generator of $\mathbb{R}^k\backslash\{0\}$, and that $j^*(c^+_{i,n})$ is the image of a generator of $\mathbb{R}^k\backslash\{q_i\}$ for $1 \le i \le n-1$, respectively. From the third diagram we see that $j^*(c^-_{i,n}) = (-1)^k\varphi^*j^*(c^+_{i,n})$ (using Lemma 7 ($\dot{w}$)), and thus in turn is the image of a generator of $\mathbb{R}^k\backslash\{\varphi(q_i)\}$. We can conclude that the classes $j^*(c_n)$, $j^*(c^+_{i,n})$, $j^*(c^-_{i,n})$ for $1 \le i \le n-1$ form a full set of generators for $H^{k-1}(F)$.

(2) A linear basis for the E_2-term of $E_*(\Pi_{[n-1]})$, and thus for the cohomology of $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$, consists of linear bases for both the non-trivial rows in the corresponding sequence tableau: By induction, a linear basis for $H^*(B)$ is given by classes $c^{\epsilon_1}_{i_1,j_1} \cup \ldots \cup c^{\epsilon_t}_{i_t,j_t}$, where $1 \le t \le n-1$, $0 \le i_r < j_r \le n-1$, $\epsilon_r \in \{\pm 1\}$ for $r = 1,\ldots,t$, and $j_1 < \ldots < j_t$. Since the spectral sequence collapses in its second term, $\Pi^*_{[n-1]}$ is injective, and by Lemma 6 the corresponding classes in $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$ form a basis for $E^{*,0}_2 = \Pi^*_{[n-1]}(H^*(B))$. Taking the classes c_n, $c^+_{i,n}$ and $c^-_{i,n}$, $1 \le i \le n-1$, and multiplying the basis elements described above with one of those, yields a basis for the second non-trivial row, $E^{*,k-1}_2 \cong H^{k-1}(F) \otimes \Pi^*_{[n-1]}(H^*(B))$. This combines to a linear basis for $H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n))$ as stated. $\square$

Remark 10 We had postponed the discussion of cohomology groups for the configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^2\backslash\{0\},n)$ (compare Remark 2). Considering again the fibre bundle $\mathbb{R}^2\backslash Q^{n-1}_{\langle\varphi\rangle} \longrightarrow \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^2\backslash\{0\},n) \longrightarrow \mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^2\backslash\{0\},n-1)$, we now see, as in the previous proof, that the map induced in cohomology by the inclusion of the fibre into the total space is surjective. Note that we do not use any information on the collapsing of the associated spectral sequence for that; we rather *conclude* that the coefficient system is simple and that the associated spectral sequence collapses in its second term [Bo, Thm. 14.1]. Continuing as in the proof of Proposition 1 we derive that

$$H^*(\,\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^2\backslash\{0\},n)\,,\,\mathbb{Z}\,)\;\cong\;\bigotimes_{j=1}^{n}(\,\mathbb{Z}(0)\,\oplus\,\mathbb{Z}(1)^{\,2j-1}\,)$$

as a graded group.

Moreover, the assertions of Proposition 8 hold for $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^2\backslash\{0\},n)$: having the triviality of coefficient systems at hand for $k=2$, the proof can be carried out analogously.

We will see that all the relations in the cohomology of $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n)$ occur already in 3-point configuration spaces, and can be transfered to the general case by Lemma 6. But the most interesting and difficult relations, i. e., those which, in a certain sense, are furthest from the arrangement case, already occur on 2-point configuration spaces! For them it is crucial to understand the geometry, and thus generators and relations for the cohomology, of the configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},2)$. Explicitly, we are thus studying the space

$$\{(x,y)\in\mathbb{R}^k\times\mathbb{R}^k\,:\,x\neq 0,\;y\neq 0,\;x\neq y,\;x\neq\varphi(y)\}.$$

This is the complement of an arrangement of four "subspaces" of $\mathbb{R}^{2k}$ of codimension k. The most interesting relation – the last one in the following listing – is the one which is supported on the complement of the subarrangement consisting of the first, third and fourth subspace.

Proposition 11 *The following relations hold in $H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\},n))$:*

$$\begin{array}{llll}
c^+_{i,l}\cup c^+_{j,l} + (-1)^{k+1}\,c^+_{i,j}\cup c^+_{j,l} + & c^+_{i,j}\cup c^+_{i,l} & = 0\,, & i<j<l\,,\\
c^-_{i,l}\cup c^+_{j,l} + (-1)^{k+1}\,c^-_{i,j}\cup c^+_{j,l} + & c^-_{i,j}\cup c^-_{i,l} & = 0\,, & i<j<l\,,\\
c^+_{i,l}\cup c^-_{j,l} - c^-_{i,j}\cup c^-_{j,l} + (-1)^k & c^-_{i,j}\cup c^+_{i,l} & = 0\,, & i<j<l\,,\\
c^-_{i,l}\cup c^-_{j,l} - c^+_{i,j}\cup c^-_{j,l} + (-1)^k & c^+_{i,j}\cup c^-_{i,l} & = 0\,, & i<j<l\,,\\
c_j\cup c^+_{i,j} + (-1)^{k+1}\,c_i\cup c^+_{i,j} + & c_i\cup c_j & = 0\,, & i<j\,,\\
c_j\cup c^-_{i,j} - c_i\cup c^-_{i,j} + (-1)^k & c_i\cup c_j & = 0\,, & i<j\,,\\
c^-_{i,j}\cup c^+_{i,j} + (-1)^{k+1}\,c_i\cup c^+_{i,j} + & c_i\cup c^-_{i,j} & = 0\,, & i<j\,.
\end{array}$$

PROOF. The first relation holds by restriction from the complement of subspaces $U^+_{i,j}$, $U^+_{i,l}$ and $U^+_{j,l}$. Observe that these subspaces are contained in the codimension k version of the Coxeter arrangement of type A, and the relation of corresponding cohomology classes holds [FZ2, Prop. 3.1].

The following three relations are obtained by applying the maps induced by the involutions A_i, A_j and A_l, respectively, to the first relation. Observe that the change of signs is a consequence of Lemma 7. The relation involving c_i, c_j and $c_{i,j}^+$ again follows by restriction from the complement of U_i, U_j and $U_{i,j}^+$, and the next relation by applying A_i^* or A_j^*.

Only the last relation requires some more work: We are concerned with the space $\mathcal{M}(\{U_1, U_{1,2}^+, U_{1,2}^-\}) \subseteq \mathbb{R}^k \times \mathbb{R}^k$ – an arrangement complement that contains the configuration space which we explicitly described prior to Proposition 11. We will show that the proposed relation holds for the cohomology generators c_1, $c_{1,2}^+$ and $c_{1,2}^- = A_1^*(c_{1,2}^+)$, where the first two are the standard generators corresponding to subspaces U_1 and $U_{1,2}^+$, respectively. As our (sloppy) notation suggests, such relation then transfers by restriction of cohomology classes to a corresponding relation for the configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \setminus \{0\}, 2)$. The relation as stated above is then a consequence of Lemma 6.

Consider the map

$$\begin{aligned} \Phi : \quad \mathcal{M}(\{U_1, U_{1,2}^+, U_{1,2}^-\}) &\longrightarrow \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) \\ (x\,,\, y) \qquad &\longmapsto (x-y\,,\, \varphi(x)-y)\,. \end{aligned}$$

For easier distinction, we denote the standard cohomology generators for the arrangement complement on the right-hand side by d_1, d_2, and $d_{1,2}$. We will show that

$$(i)\ \ \Phi^*(d_1) \;=\; c_{1,2}^+\,, \qquad (ii)\ \ \Phi^*(d_2) \;=\; c_{1,2}^-\,, \qquad (iii)\ \ \Phi^*(d_{1,2}) \;=\; c_1\,,$$

and thus transfer the usual 3-term relation in the arrangement complement to the stated relation involving c_1, $c_{1,2}^+$ and $c_{1,2}^-$.

(i) Let $\Pi_1 : \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) \longrightarrow \mathbb{R}^k \setminus \{0\}$ denote the projection to the first coordinate, and $\theta : \mathcal{M}(\{U_1, U_{1,2}^+, U_{1,2}^-\}) \longrightarrow \mathbb{R}^k \setminus \{0\}$ the map given by $\theta(x, y) = x-y$. Obviously, $\Pi_1 \circ \Phi = \theta$. Let c denote the standard generator for $\mathbb{R}^k \setminus \{0\}$. By analogy to the statements of Lemma 6, $\Pi_1^*(c) = d_1$, and moreover, $\theta^*(c) = c_{1,2}^+$. Thus, we conclude that

$$\Phi^*(d_1) \;=\; \Phi^*\Pi_1^*(c) \;=\; \theta^*(c) \;=\; c_{1,2}^+\,.$$

(ii) With τ denoting transposition of coordinates, observe that $\Phi \circ A_1 = \tau \circ \Phi$. We conclude that

$$\Phi^*(d_2) \;=\; \Phi^*\tau^*(d_1) \;=\; A_1^*\Phi^*(d_1) \;=\; A_1^*(c_{1,2}^+) \;=\; c_{1,2}^-\,.$$

(*iii*) Let $\sigma : \mathcal{M}(\{U_1, U_{1,2}^+, U_{1,2}^-\}) \longrightarrow S^{k-1}$ denote projection to the first coordinate and further to the unit sphere in $\mathbb{R}^k$, and let $\bar{\theta} : \mathcal{M}(\{U_1, U_2, U_{1,2}^+\}) \longrightarrow S^{k-1}$ be given by $\bar{\theta}(x,y) = \|x-y\|^{-1}(x-y)$. With $\varphi(-x) = -\varphi(x)$ we see that

$$\begin{aligned}\bar{\theta}(\Phi(x,y)) &= \bar{\theta}(x-y, \varphi(x)-y) \\ &= \frac{1}{\|x-\varphi(x)\|}\,(x-\varphi(x)) \\ &= \frac{1}{(1+\frac{1}{\|x\|^2})\|x\|}\,(1+\tfrac{1}{\|x\|^2})\,x \;=\; \sigma(x,y)\,.\end{aligned}$$

Let now c be the standard generator of $\mathbb{R}^k\backslash\{0\}$ restricted to S^{k-1}. Analogous to the situation in (i), we see that $\sigma^*(c) = c_1$, and $\bar{\theta}^*(c) = d_{1,2}$, and we conclude that

$$\Phi^*(d_{1,2}) \;=\; \Phi^*\bar{\theta}^*(c) \;=\; \sigma^*(c) \;=\; c_1\,.$$

The last relation of our proposition follows as described above. $\square$

Theorem 12 *For $k \geq 2$, $n \geq 1$, the integer cohomology algebra of the orbit configuration space $\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n)$ can be presented as a quotient of an exterior algebra generated in degree $k-1$ by generators corresponding to the cohomology classes c_i, $1 \leq i \leq n$, and $c_{i,j}^+$, $c_{i,j}^-$, $1 \leq i < j \leq n$:*

$$H^*(\,\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n)\,,\,\mathbb{Z}\,) \;\cong\; \Lambda^*\;(\,\mathbb{Z}(k-1)^{n+2\binom{n}{2}}\,)\;\big/\;J\,,$$

where the ideal J is generated by the relations listed in Proposition 11.

PROOF. We can adopt terminology and proof techniques from the arrangement context: our basis elements described in Proposition 8(2) are indexed with elements of the broken circuit complex associated with the Coxeter arrangement of type B (cf. Remark 9) and the relations described in Proposition 11 are indexed with the circuits of the same arrangement. A standard straightening argument (compare [Bj1, Lemma 7.10.1(ii)] for the arrangement case) shows that our relations allow each product in the generators c_i, $c_{i,j}^+$, $c_{i,j}^-$ to be written as a linear combination of the basis elements of Proposition 8(2), which completes our proof of the stated algebra description. $\square$

4 The spectral sequence

We get back to the fibre map on the orbit configuration space $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ that we described in Section 2. With the results of the previous section we now have all information at hand to study the associated Leray-Serre spectral sequence $E_*(\Pi)$.

With the base space S^k being simply connected for $k \geq 2$ and its cohomology being torsion free, the E_2-term of the spectral sequence takes the simple form

$$E_2^{p,q} \;\cong\; H^p(S^k) \,\otimes\, H^q(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \backslash \{0\}, n-1))\,, \qquad p, q \geq 0\,.$$

The only possibly non-trivial differential occurs on the E_k-term and has bidegree $(k, -k+1)$. By Proposition 8(1) it is enough to determine its action on $E_k^{0,k-1}$:

Proposition 13 *The k-th differential d^k acts as follows on the multiplicative generators of $E_k^{0,*}(\Pi) \;\cong\; H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \backslash \{0\}, n-1))$:*

$$d^k(c_i) \;=\; d^k(c_{i,j}^+) \;=\; d^k(c_{i,j}^-) \;=\; \begin{cases} 0 & \text{for odd } k, \\ 2\,\nu(S^k) & \text{for even } k, \end{cases}$$

with $1 \leq i \leq n-1$, and $1 \leq i < j \leq n-1$, respectively, where $\nu(S^k)$ denotes an orientation class for S^k.

PROOF. We closely follow [FZ2, Sect. 4.2], where we described the differential of a similar spectral sequence associated with ordered configuration spaces of spheres. Here we content ourselves with an outline.

First note that the S_{n-1}-action on $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ given by permutation of the first $n-1$ coordinates of a configuration induces a group action on the fibre bundle and thus on the spectral sequence. S_{n-1} acts transitively on both the sets of generators $\{c_i\}_{1\leq i\leq n-1}$ and $\{c_{i,j}^+\}_{1\leq i<j\leq n-1}$ of $E_k^{0,*}(\Pi)$. For these sets of generators, it is thus sufficient to determine the differential on c_1 and on $c_{1,2}^+$. Consider the following maps of fibre bundles given by inclusions:

$$\begin{array}{ccccc}
\mathcal{M}(\{U_1\}) & \longrightarrow & \{x \in (S^k)^n \,|\, x_i \neq x_n,\, 1 \leq i \leq n-1,\, x_1 \neq -x_n\} & \longrightarrow & S^k \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \backslash \{0\}, n-1) & \longrightarrow & \mathrm{F}_{\mathbb{Z}_2}(S^k, n) & \longrightarrow & S^k\,,
\end{array}$$

$$
\begin{array}{ccccc}
\mathcal{M}(\{U^+_{1,2}\}) & \longrightarrow & \{x \in (S^k)^n \,|\, x_i \neq x_n,\, 1 \leq i \leq n-1,\, x_1 \neq x_2\} & \longrightarrow & S^k \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \setminus \{0\}, n-1) & \longrightarrow & \mathrm{F}_{\mathbb{Z}_2}(S^k, n) & \longrightarrow & S^k\,.
\end{array}
$$

The fibre bundles in the upper rows are fibre homotopy equivalent to the unit tangent bundle on S^k. The only non-trivial differentials in the associated spectral sequences map the single generators for the cohomology of the fibres, $\hat{c}_1$ and $\hat{c}_{1,2}$, respectively, to the Euler class of the base space S^k. By the induced maps of spectral sequences this transfers to the cohomology classes c_1 and $c^+_{1,2}$ in $E^{0,k-1}_k(\Pi)$.

Similarly, the action of d^k on $c^+_{i,j}$ determines the action of d^k on $c^-_{i,j}$. The map of fibre bundles which transfers the cohomology generators of the fibres appropriately, is given by applying the antipodal map on S^k to the i-th point of a configuration and keeping all other points fixed. $\square$

For odd k, the spectral sequence $E_*(\Pi)$ thus collapses in its second term and we can formulate the complete result on the cohomology algebra of $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ in that case:

Theorem 14 *For odd k, $k \geq 3$, $n \geq 1$, the integer cohomology algebra of the orbit configuration space $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ can be described as*

$$
\begin{aligned}
H^*(\,\mathrm{F}_{\mathbb{Z}_2}(S^k, n)\,,\,\mathbb{Z}\,) \;&\cong\; (\,\mathbb{Z}(0) \oplus \mathbb{Z}(k)\,) \otimes H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \setminus \{0\}, n-1)) \\
&\cong\; (\,\mathbb{Z}(0) \oplus \mathbb{Z}(k)\,) \otimes \Lambda^*\,(\,\mathbb{Z}(k-1)^{n-1+2\binom{n-1}{2}}\,) \Big/ J\,,
\end{aligned}
$$

where the ideal J is generated by the relations listed in Proposition 11.

It remains to determine the E_{k+1}-term of the spectral sequence $E_*(\Pi)$ for even k, and to reconstruct the cohomology algebra of $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ from it. We can follow almost word by word our corresponding discussion for ordered configuration spaces of spheres in [FZ2, Sect. 5]. Though, in the present situation, the fibre is *not* a complement of a linear subspace arrangement, the “arrangement-like” properties of its cohomology derived in Section 3 suffice to completely parallel the arguments used in the case of ordinary configuration spaces.

Proposition 15 *For $0 \le i < j \le n{-}1$, $j \neq 1$, and $\epsilon \in \{\pm 1\}$, set $e^{\epsilon}_{i,j} := c^{\epsilon}_{i,j} - c_1$ in $H^{k-1}(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \backslash \{0\}, n{-}1))$. Then*

$$\mathcal{B} := \{c_1 \cup e_{\alpha_1} \cup \ldots \cup e_{\alpha_t} \mid \{\alpha_1, \ldots, \alpha_t\} \in \mathrm{BC}(\mathcal{B}^{(k)}_{n-1}),\ \alpha_i \neq (0,1,\pm 1)\}$$
$$\cup \{e_{\alpha_1} \cup \ldots \cup e_{\alpha_t} \mid \{\alpha_1, \ldots, \alpha_t\} \in \mathrm{BC}(\mathcal{B}^{(k)}_{n-1}),\ \alpha_i \neq (0,1,\pm 1)\}$$

is a $\mathbb{Z}$-linear basis for $H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \backslash \{0\}, n{-}1))$; here the α_i stand for index triples (i, j, ϵ), $0 \le i < j \le n{-}1$, $j \neq 1$.

Let $\mathcal{T}^{\bullet}$ denote the submodule of $H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \backslash \{0\}, n{-}1))$ generated by those elements of $\mathcal{B}$ that contain c_1 as a factor, and let $\mathcal{T}^{\circ}$ be the submodule generated by the other elements of $\mathcal{B}$.

Proposition 16 *$\mathcal{T}^{\circ}$ is a subalgebra of $H^*(\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k \backslash \{0\}, n{-}1))$ generated by the cohomology classes $e^{\epsilon}_{i,j}$ for $0 \le i < j \le n{-}1$, $j \neq 1$, and $\epsilon \in \{\pm 1\}$. It has a presentation as a quotient of the exterior algebra on these generators:*

$$\mathcal{T}^{\circ} \;\cong\; \Lambda^* \left(\mathbb{Z}(k{-}1)^{n-2+2\binom{n-1}{2}} \right) \Big/ \, I\,,$$

where I is the ideal generated by the elements

$$\begin{array}{llllr}
e^{+}_{i,l} \wedge e^{+}_{j,l} & +\ (-1)^{k+1}\, e^{+}_{i,j} \wedge e^{+}_{j,l} & + & e^{+}_{i,j} \wedge e^{+}_{i,l}\,, & i < j < l\,, \\
e^{-}_{i,l} \wedge e^{+}_{j,l} & +\ (-1)^{k+1}\, e^{-}_{i,j} \wedge e^{+}_{j,l} & + & e^{-}_{i,j} \wedge e^{-}_{i,l}\,, & i < j < l\,, \\
e^{+}_{i,l} \wedge e^{-}_{j,l} & -\ e^{-}_{i,j} \wedge e^{-}_{j,l} & +\ (-1)^{k} & e^{-}_{i,j} \wedge e^{+}_{i,l}\,, & i < j < l\,, \\
e^{-}_{i,l} \wedge e^{-}_{j,l} & -\ e^{+}_{i,j} \wedge e^{-}_{j,l} & +\ (-1)^{k} & e^{+}_{i,j} \wedge e^{-}_{i,l}\,, & i < j < l\,, \\
e_{j} \wedge e^{+}_{i,j} & +\ (-1)^{k+1}\ e_{i} \wedge e^{+}_{i,j} & + & e_{i} \wedge e_{j}\,, & 1 < i < j\,, \\
e_{j} \wedge e^{-}_{i,j} & -\ e_{i} \wedge e^{-}_{i,j} & +\ (-1)^{k} & e_{i} \wedge e_{j}\,, & 1 < i < j\,, \\
e^{-}_{i,j} \wedge e^{+}_{i,j} & +\ (-1)^{k+1}\ e_{i} \wedge e^{+}_{i,j} & + & e_{i} \wedge e^{-}_{i,j}\,, & 1 < i < j\,, \\
e_{j} \wedge e^{+}_{1,j}\,, & e_{j} \wedge e^{-}_{1,j}\,, & & e^{-}_{1,j} \wedge e^{+}_{1,j}\,, & 2 < j\,.
\end{array}$$

The columns of the $(k{+}1)$-th sequence tableau can now be described as

$$\begin{aligned}
E^{0,*}_{k+1} = \ker d^k &\;=\; \mathcal{T}^{\circ} \\
E^{k,*}_{k+1} = \operatorname{coker} d^k &\;=\; \mathcal{T}^{\circ}/2\mathcal{T}^{\circ} \,\oplus\, \mathcal{T}^{\bullet}\,,
\end{aligned}$$

where $E^{0,*}_{k+1}$ is multiplicatively generated in $E^{0,k-1}_{k+1}$ by the classes $e^{\epsilon}_{i,j}$ for

$0 \le i < j \le n-1$, $j \ne 1$, $\epsilon \in \{\pm 1\}$, and $E_{k+1}^{k,*}$ is generated by the free generators in $E_{k+1}^{0,k-1}$, the free generator c_1 in $E_{k+1}^{k,k-1}$, and the generator of order 2 in $E_{k+1}^{k,0}$.

As a graded group, $H^*(\mathrm{F}_{\mathbb{Z}_2}(S^k, n))$ can be reconstructed from the E_{k+1}-term as the direct sum of the two non-trivial columns in the corresponding sequence tableau. For $k > 2$, this is a trivial consequence of each diagonal having only one non-trivial entry. For $k = 2$, the claim follows since

$$E_3^{0,*} \;\cong\; H^*(\mathrm{F}_{\mathbb{Z}_2}(S^2, n)) \,/\, E_3^{2,*-2}\,,$$

and $E_3^{0,*}$ is torsion-free.

We deduce the following algebra description for $H^*(\mathrm{F}_{\mathbb{Z}_2}(S^k, n))$ from the multiplicative structure of the E_{k+1}-term as we described it above:

Theorem 17 *For even $k \ge 2$, and $n \ge 1$, the integer cohomology algebra of the orbit configuration space $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ can be described as*

$$\begin{aligned} H^*(\,\mathrm{F}_{\mathbb{Z}_2}(S^k, n)\,,\,\mathbb{Z}\,) \;&\cong\; (\,\mathbb{Z}(0) \oplus \mathbb{Z}_2(k) \oplus \mathbb{Z}(2k-1)\,) \,\otimes\, \ker d^k \\ &\cong (\,\mathbb{Z}(0) \oplus \mathbb{Z}_2(k) \oplus \mathbb{Z}(2k-1)\,) \,\otimes\, \Lambda^*\,\left(\,\mathbb{Z}(k-1)^{n-2+2\binom{n-1}{2}}\,\right) \Big/\, I\,, \end{aligned}$$

where I is the ideal of relations described in Proposition 16*.*

We claimed in the Introduction that the orbit configuration spaces $\mathrm{F}_{\mathbb{Z}_2}(S^k, n)$ can be seen as "type B" configuration spaces of k-spheres. This naturally assigns the role of "type A" configuration spaces to the classical ordered configuration spaces of spheres $\mathrm{F}(S^k, n)$.

Indeed, the essential ingredient for our computation of the cohomology of $\mathrm{F}(S^k, n)$ in [FZ2] was the Fadell-Neuwirth fibre bundle:

$$\mathrm{F}(\mathbb{R}^k, n-1) \;\longrightarrow\; \mathrm{F}(S^k, n) \;\longrightarrow\; S^k\,.$$

The fibre is the complement of a codimension k version of the arrangement of reflecting hyperplanes associated with the Coxeter group of type A:

$$\mathrm{F}(\mathbb{R}^k, n-1) \;=\; \mathcal{M}(\mathcal{A}_{n-2}^{(k)})\,,$$

where the arrangement $\mathcal{A}_{n-2}^{(k)}$ is formed by the linear subspaces

$$W_{i,j} \;:=\; \{\,(x_1, \ldots, x_{n-1}) \in (\mathbb{R}^k)^{n-1} \,|\, x_i = x_j\,\}\,, \qquad 1 \le i < j \le n-1\,.$$

Essential ingredient in the present studies is the fibre bundle

$$\mathrm{F}_{\langle\varphi\rangle}(\mathbb{R}^k\backslash\{0\}, n-1) \longrightarrow \mathrm{F}_{\mathbb{Z}_2}(S^k, n) \longrightarrow S^k\,,$$

where the fibre can be viewed as a (non-linear) version of the subspace arrangement $\mathcal{B}_{n-1}^{(k)}$. It shares with the latter the description of its cohomology as a graded group (Proposition 3), the combinatorial index set of a linear basis (Remark 9), and it has a description for its cohomology algebra of the same flavour as it is standard for type B arrangements (Theorem 12).

In both cases, the cohomology algebras of the fibres figure in the descriptions of the cohomology algebras of the configuration spaces: Most visibly they appear for odd k, namely as factors in a product decomposition that stems from the (final!) E_2-term of the cohomological Leray-Serre spectral sequence for the respective fibre bundles. For even k, a closely related subalgebra – the kernel of the only non-trivial differential in the spectral sequence – occurs as a factor of the cohomology algebras.

This close relation to the respective arrangement complements gives reason enough to talk about configuration spaces of spheres of Coxeter type.

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