Shellability of complexes of trees

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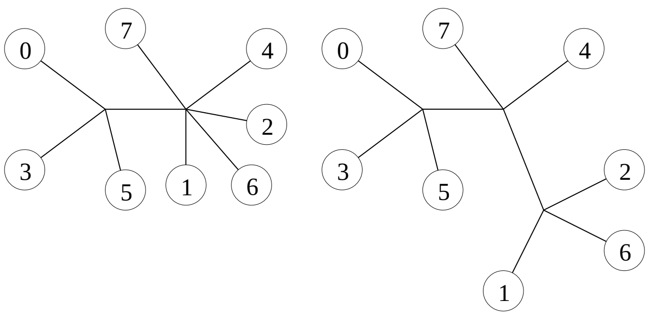
We show that for all $k \geq 1$ and $n \geq 0$ the simplicial complexes $\mathcal{T}_n^{(k)}$ of all leaf-labelled trees with $nk+2$ leaves and all interior vertices of degrees $kl+2$ $(l \geq 1)$ are shellable. This yields a direct combinatorial proof that they are Cohen-Macaulay and that their homotopy types are wedges of spheres.

Introduction.

A very interesting abstract simplicial complex $\mathcal{T}_n^{(k)}$ has faces in bijection with the trees with at most n interior vertices, all of which have degrees at least $k+2$ and congruent to 2 mod k, and whose leaves are labelled by the distinct integers in $\{0, 1, \ldots, m\}$, where $m+1 := nk+2$ is the number of leaves ($n \geq 0$, $k \geq 1$). Thus the facets of $\mathcal{T}_n^{(k)}$ correspond to the leaf-labelled trees with n interior vertices of degree exactly $k+2$, while the vertices of the complex correspond to the trees with exactly one interior edge, and two internal nodes of degrees $kl+2$ and $k(n-l)+2$, with $1 \leq l \leq n-1$. The partial order on these trees that is induced by contraction of interior edges corresponds to inclusion relation between faces of the complex $\mathcal{T}_n^{(k)}$. The complex $\mathcal{T}_n^{(k)}$ has $\sum_{i=1}^{n-1} \binom{m}{ki+1}$ vertices. Its dimension is $n-2$.

For example, for $n = 3$ and $k = 2$ we obtain a 1-dimensional simplicial complex (i.e., a graph) with $\binom{7}{3} + \binom{7}{5} = 56$ vertices corresponding to graphs with one interior edge as depicted on the left of our picture, and $\frac{1}{2}\binom{8}{2}\binom{6}{3}$ facets (graph edges) as depicted on the right.



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For $k = 1$, the complex $\mathcal{T}_n^{(1)}$ triangulates the "space of fully grown trees" of Boardman [5]; see Adin & Blanc [1] for a recent appearance of this space in a homotopy theory setting.

From a representation theory point of view, the complex $\mathcal{T}_n^{(k)}$ has an interesting action of $\mathfrak{S}_{m+1}$, which induces an interesting representation of $\mathfrak{S}_{m+1}$ on the homology of $\mathcal{T}_n^{(k)}$. For this purpose it was determined that

- for $k = 1$, the complex $\mathcal{T}_n^{(1)}$ has the homotopy type of a wedge of $n!$ $(n-2)$-spheres (Robinson [8, 9]).
- also for $k > 1$, the spaces $\mathcal{T}_n^{(k)}$ are Cohen-Macaulay; Hanlon's proof [6] has two parts:
  (i) all the links in a tree complex are themselves joins of tree complexes, and
  (ii) $\mathcal{T}_n^{(k)}$ has the homotopy type of a wedge of $(n-2)$-spheres: Robinson's topological argument can be extended to the case $k > 1$, according to J.-L. Loday (unpublished).

In this context a combinatorial argument for the shellability of the simplicial complexes $\mathcal{T}_n^{(k)}$ is desirable (see [6, p. 305]!), since from this one obtains

- the homotopy type (as a wedge of spheres),
- the Cohen-Macaulay property (over $\mathbb{Z}$),
- and the homology (whose rank is the number of spheres in the wedge, i.e., the dimension of the representations studied).

In this note we provide a shellability proof.

(Note: Hanlon [6] works with the order complex $\Delta(\mathcal{L}_n^{(k)})$ of the face lattice $\mathcal{L}_n^{(k)}$ of $\mathcal{T}_n^{(k)}$, which is the barycentric subdivision of the complex $\mathcal{T}_n^{(k)}$ that we study in this paper. Thus shellability of $\mathcal{T}_n^{(k)}$ implies "dual CL" shellability, cf. [4], of Hanlon's complex $\Delta(\mathcal{L}_n^{(k)})$. It also implies Cohen-Macaulayness of $\mathcal{T}_n^{(k)}$, which is equivalent to that of $\Delta(\mathcal{L}_n^{(k)})$.)

Additionally we obtain, in the last section, an explicit set of $\beta_n^{(k)}$ facets that yields a basis for the (co)homology of the complex $\mathcal{T}_n^{(k)}$. This basis is equivalent to the basis constructed by Hanlon & Wachs [7, Sect. 2] for the multiplicity-free part $F[\mathbf{1}]$ of the free Lie k-algebra. (With hindsight, one might perhaps have guessed the correct way to shell $\mathcal{T}_n^{(k)}$ from the constructions of [7, p. 218]?)

For small n and for small k, we derive explicit formulas for the dimensions $\beta_n^{(k)}$:

$$\beta_n^{(1)} = n! \qquad \beta_n^{(2)} = \left(\frac{(2n)!}{2^n n!}\right)^2 \qquad \beta_1^{(k)} = 1 \qquad \beta_2^{(k)} = \binom{2k+1}{k} - 1.$$

Reverse lexicographic order.

For the following $k \geq 1$ and $n \geq 0$ are fixed integers. We use the notation $[n]$ for $\{1, 2, \ldots, n\}$. The symbol $\subset$ denotes strict inclusion of (finite) sets. The set of all subsets of V is written as 2^V, while $\binom{V}{r}$ is the collection of all r-element subsets of V. On finite sets (of integers), we use $\prec$ to denote the *reverse lexicographic* total order defined by

$$A \prec B \quad :\Longleftrightarrow \quad \max((A \backslash B) \cup (B \backslash A)) \in B.$$

We will use only two (obvious) properties of this order:

$$\begin{aligned} A \subset B \quad &\Longrightarrow \quad A \prec B \\ \max(A) < \max(B) \quad &\Longrightarrow \quad A \prec B, \end{aligned}$$

so any other order that satisfies these two properties would also be fine for our purposes.

Simplicial complexes and shellings.

All the complexes that we consider are finite, abstract, pure simplicial complexes represented by their collections of facets.

Definition 1 Let $\mathcal{C}$ be a pure simplicial complex (given by a finite collection of finite sets of the same cardinality, the *facets* of $\mathcal{C}$).

A *shelling* of $\mathcal{C}$ is a linear order "$<$" on the set of facets such that for any two facets $C' < C$ there is some facet C'' of the complex as well as an element $x \in C$ such that
(S1) $C'' < C$,
(S2) $x \notin C'$, and
(S3) $C \backslash x \subseteq C''$.

The three conditions of this definition imply that

$$C' \cap C = (C' \backslash x) \cap C = C' \cap (C \backslash x) \subseteq C' \cap C'' \subseteq C''$$

and hence
(S1*) $C' \cap C \subseteq C'' \cap C$,
(S2*) $C'' < C$, and
(S3*) C'' differs from C in only one element, $C'' \backslash C = \{x\}$,
which are the conditions that are usually used to define shellings [3, 4]. Conversely, if we have $C'' < C$ such that $C' \cap C \subseteq C'' \cap C$ and $C'' \backslash C = \{x\}$, then the conditions (S1) to (S3) are also satisfied.

Leaf-labelled trees.

Let T be a *k-tree of size n*: a tree with n interior (non-leaf) vertices, each of degree exactly $k+2$. Such a tree has $n-1$ interior edges and $nk+2$ leaf edges. Our trees are *leaf-labelled*: their

$$m+1 \ := \ nk+2$$

leaf vertices (of degree 1) are injectively labelled by nonnegative integers, where one leaf must have the label 0.

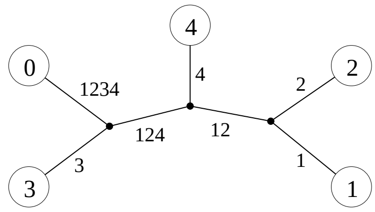
We associate with every edge e of T the set $l(e)$ of labels of all the leaves that e separates from the leaf labelled 0. Thus $l(e)$ is a subset of M. By $\widehat{L}(T)$ we denote the set of all edge labels of T: this includes the sets $\{i\}$ $(i \in M)$ and M of sizes 1 or m associated to the leaf edges, as well as the $n-1$ sets $l(e)$ of sizes $1 < |l(e)| < m$ associated to the interior edges of T. Let $L(T)$ be the collection of label sets of interior edges, such that

$$\widehat{L}(T) \quad = \quad L(T) \ \uplus \ \{\{i\} : i \in M\} \cup \{M\}.$$

In the following, $\mathcal{T}^{(k)}(M)$ denotes the (finite) set of all k-trees of size n whose set of leaf-labels is $\{0\} \uplus M$. Thus, in particular, $\mathcal{T}_n^{(k)} := \mathcal{T}^{(k)}([m])$ is the abstract simplicial complex described in the introduction.

Our next figure shows an example tree for $k=1$ and $n=3$, with $m+1=5$ leaves. Its label sets are $L(T) = \{\{1,2,4\},\{1,2\}\}$ and $\widehat{L}(T) = \{\{1\},\{2\},\{3\},\{4\},\{1,2,3,4\},\{1,2,4\},\{1,2\}\}$.

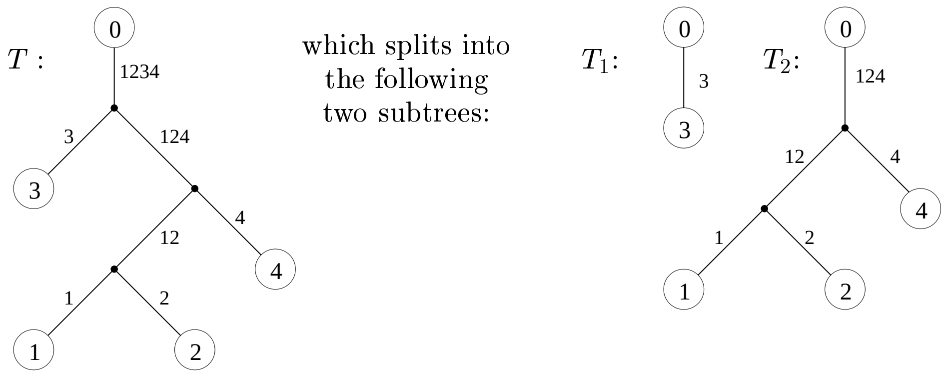
In the figure the edge labels are shown without set brackets:



The edge labels of a leaf-labelled k-tree allow one to reconstruct the tree uniquely — this is an important observation that allows us to describe and handle trees in terms of (only) their label sets.

Every k-tree with more than one edge can be decomposed into $k+1$ subtrees, which are trees of their own: If $M_0, \ldots, M_k$ are the (disjoint!) maximal sets in $\widehat{L}(T)\backslash\{M\}$, then the subtrees are given by $\widehat{L}(T_i) = \{N \in \widehat{L}(T) : N \subseteq M_i\} = \widehat{L}(T) \cap 2^{M_i}$. We will always order the $k+1$ subtrees by using reverse lexicographic order on their labels sets, that is, the subtrees $T_0, \ldots, T_k$ are named such that their label sets $M_0, M_1, \ldots, M_k$ satisfy $M_0 \prec \ldots \prec M_k$.

Our next figure displays the tree (with $M = [4]$) that we have looked at before. It is now displayed with the leaf labelled 0 as the root at the top, and with the $k+1$ subtrees at each interior node displayed left-to-right (here we have $k = 1$, with $M_0 = \{3\}$ and $M_1 = \{1, 2, 4\}$):



Tree complexes.

By $\widehat{\mathcal{T}}^{(k)}(M)$ we denote the complex of edge label sets of k-trees with label set $\{0\} \cup M$, while by $\mathcal{T}^{(k)}(M)$ we denote the complex of interior label sets of k-trees with label set $\{0\} \cup M$:

$$\begin{aligned} \mathcal{T}^{(k)}(M) &:= \{L(T) : T \text{ is a } k\text{-tree with leaf-labels } \{0\} \cup M\} \\ \widehat{\mathcal{T}}^{(k)}(M) &:= \{\widehat{L}(T) : T \text{ is a } k\text{-tree with leaf-labels } \{0\} \cup M\}. \end{aligned}$$

Deletion of label sets from $L(T)$ corresponds to contraction of interior edges of T. Thus the faces of the complex $\mathcal{T}^{(k)}(M)$ can be identified with the set of all leaf-labelled trees with label set $\{0\} \cup M$ and with all vertex degrees $\equiv 2 \bmod k$, ordered by contraction.

Since the label sets of leaf edges are the same for all trees with the same label set $\{0\} \cup M$, we find that the complex $\widehat{\mathcal{T}}^{(k)}(M)$ is just a multiple cone over the complex $\mathcal{T}^{(k)}(M)$.

$N \subseteq M$ can occur as an edge label for a tree in $\widehat{\mathcal{T}}^{(k)}(M)$ if and only if $|N| \equiv 1 \bmod k$. Thus $\widehat{\mathcal{T}}^{(k)}(M)$ is a simplicial complex of dimension $n(k+1)$ on $\sum_{i \geq 0} \binom{m}{ki+1}$ vertices. The complex $\mathcal{T}^{(k)}(M)$ has $m+1$ vertices less, but only dimension $n(k+1) - (m+1) = n - 2$.

Theorem 2 *For any $k \geq 1$, $n \geq 1$ and any label set $M \subseteq \mathbb{N}$ of size $m = nk + 1$, the set families $\mathcal{T}^{(k)}(M)$ and $\widehat{\mathcal{T}}^{(k)}(M)$ are the facet systems of shellable simplicial complexes.*

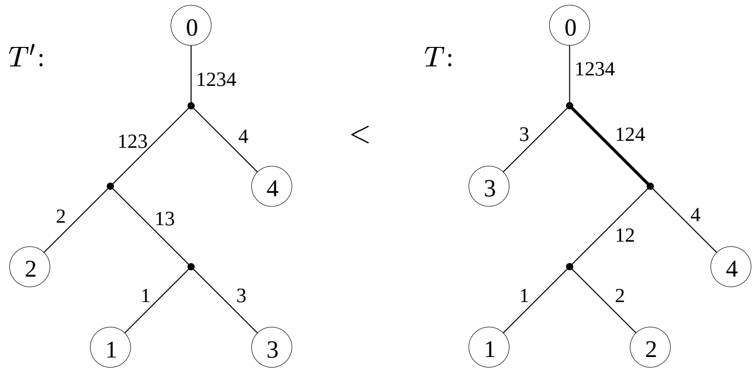
Cone vertices are irrelevant for shellings, so $\widehat{\mathcal{T}}^{(k)}(M)$ is shellable if and only if $\mathcal{T}^{(k)}(M)$ is shellable. For convenience we work with the complex $\widehat{\mathcal{T}}^{(k)}(M)$ when proving Theorem 2 in the following.

Shelling.

Now we simplify the notation by identifying each tree with its set of labels, that is, by writing T instead of $\widehat{L}(T)$.

Definition 3 The linear order "$<$" on $\widehat{\mathcal{T}}^{(k)}(M)$ is trivial on $\widehat{\mathcal{T}}^{(k)}(\{i\})$. For $|M| > 1$ and different trees $T', T \in \mathcal{T}^{(k)}(M)$, let $T'_0, \ldots, T'_k$ and $T_0, \ldots, T_k$ denote the corresponding subtrees. We define recursively: $T' < T \ :\Longleftrightarrow \begin{cases} M'_j \prec M_j & \text{or} \\ M'_j = M_j & \text{and } \ T'_j < T_j, \end{cases}$
where $j := \max\{i : T'_i \neq T_i\}$ is the index of the rightmost subtree in which T and T' differ.

Our example shows two trees $T', T \in \widehat{\mathcal{T}}^{(k)}([4])$ with $k = 1$. We have $j = 1$ with $M'_1 = \{4\} \prec \{1,2,4\} = M_1$, and hence $T' < T$:



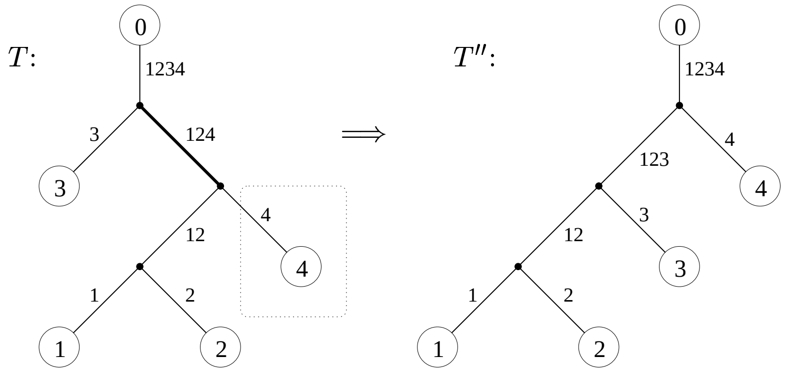
Theorem 4 *For all $k \geq 1$ and $n \geq 1$, the linear order $<$ is a shelling order for $\widehat{\mathcal{T}}^{(k)}(M)$.*

Proof. For $|M| = 1$ this is trivial. Thus we assume that $T' < T$, where T' and T split into subtrees as above.
Case 1: $M'_j < M_j$**.** We first verify three claims (a)-(c).
(a) $j > 0$: This holds since $M'_0 \uplus \ldots \uplus M'_k$ and $M_0 \uplus \ldots \uplus M_k$ are partitions of the same set M.
(b) M_j is not the label of an edge of T': Otherwise we would have some i with $M_j \subseteq M'_i$. But the sets M_i are ordered by their maximal elements, so $\max(M_j) = \max(M'_j)$ by definition of j. This would imply $i = j$ and $M_j \subset M'_j$, and hence $M_j \prec M'_j$, which cannot be.
(c) In particular, we have $|M_j| > 1$.

With (a)-(c) we have verified all we need for the exchange step. From T, we will exchange the element $x := M_j$. By (c) this is not the label of a leaf edge, so T_j is composed of $k + 1$ maximal subtrees; let $T_{j:k}$ denote the right-most subtree of T_j, that is, the subtree with $\max(M_{j:k}) = \max(M_j)$.

We construct T'' from T by removing the edge label set M_j, and adding the set $M''_{j-1} := (M_j \backslash M_{j:k}) \cup M_{j-1}$. That is, the tree T'' is obtained from T by exchanging the subtree $T_{j:k}$ by the subtree T_{j-1}. This subtree exists, since we know $j > 0$, by (a). The new tree T'' will again be composed of $k+1$ subtrees, where M''_j contains the largest element of M_j, and M''_{j-1} contains (the largest element of) M_{j-1}, while $T''_i = T_i$ for $i \notin \{j, j-1\}$. This implies $M''_{j-1} \prec M''_j$, and our labelling is again "correct" in the sense that we have $M''_0 \prec \ldots \prec M''_k$.

Our next figure shows the construction of T'' from T for the above example: here $j = 1$, the subtree T_1 has label set $M_j = \{1,2,4\}$, its subtree $T_{1:1}$ (enclosed in a dotted box) with the highest label consists of just one edge, and has label set $M_{1:1} = \{4\}$, and this is exchanged for the subtree T_0, which has label set $M_0 = \{3\}$:



Now we can verify the shelling conditions. We have found a new facet T'' of our complex $\widehat{\mathcal{T}}^{(k)}(M)$, and an element $x = M_j$ of T. This element is not contained in T', by (b), so we have (S2). Condition (S3) is satisfied by construction. For (S1) we observe that $T''_i = T_i$ holds for $i > j$, while for the index j we have $M''_j \subset M_j$, implying $T'' < T$, as required.

Case 2: $M'_j = M_j$, $T'_j < T_j$.

In this case we can exchange within the subtree T_j. In fact, we have $T'_j, T_j \in \widehat{\mathcal{T}}^{(k)}(M^*)$ for $M^* := M'_j = M_j$. By induction ($|M^*| < |M|$) we get a new subtree $T''_j \in \widehat{\mathcal{T}}^{(k)}(M^*)$ which satisfies $T''_j < T_j$ and arises from T_j by a legal shelling exchange, $T''_j \backslash N''_j = T_j \backslash N_j$ with $N_j \notin T'_j$.

Using this we can define $T'' := (T \backslash \{N_j\}) \cup \{N''_j\}$. Then we have $T'' < T$ (S1): because of $M'_j = M_j$ again T''_j is the jth subtree of T''. Also we have $N_j \notin T'$ (S2), otherwise we would have $N_j \in T'_j$ because of $N_j \subseteq M_j = M'_j$. Condition (S3), $T \backslash N_j \subseteq T''$, is satisfied by construction. □

Computing the $\beta_n^{(k)}$.

Corollary 5 *The geometric realization of $\mathcal{T}^{(k)}(M)$ has the homotopy type of a wedge of $\beta_n^{(k)}$ $(n-2)$-spheres,*

$$||\mathcal{T}^{(k)}(M)|| \simeq \bigvee_{\beta_n^{(k)}} S^{n-2}, \qquad \widetilde{\chi}(\mathcal{T}^{(k)}(M)) = (-1)^n \beta_n^{(k)},$$

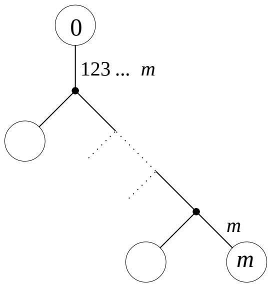
where $\beta_n^{(k)}$ is the number of k-trees with n internal nodes (with label set $[m]$) for which none of the internal edges is leftmost.

Proof. See Björner [3] [2, Sect. 7.7] and Ziegler [10, Sect. 1] for the homotopy types and the cohomology of shellable complexes. We have to identify the facets T such that for all elements (internal vertices) $M_i \in T$, there is some smaller facet $T' < T$ such that $T \backslash M_j \subseteq T'$. Now if $j > 0$, i. e. if M_j is not a leftmost edge, then we can construct $T' < T$ by replacing T_{j-1} with the largest subtree of T_j, as in the previous proof.

If $j = 0$, then a suitable $T' < T$ cannot exist: indeed, using induction we may assume that we are considering the node at the leaf with label 0, that is, $M = M_0 \cup \ldots \cup M_k$. The sets $M_0, \ldots, M_k$ label internal edges for both T and T'; no two of these labels can occur in a common subtree, since in this case we would get $T' > T$. Thus $M_1, \ldots, M_k$ label the stems of subtrees of T, and the partition property then implies $M_0 \in T'$: contradiction. □

The trees where no internal edge is leftmost appear as *k-brushes* in Hanlon & Wachs [7, Definition 2.5]. Counting them is equivalent to computing the dimension of the corresponding k-tree representation, and also to determining the dimension of the multiplicity free part $F[\mathbf{1}]$ of the free k-ary Lie algebra, by [7, Theorem 2.6]

For $\underline{k=1}$ the trees that we get this way are the "right combs" of the form



and thus $\beta_n^{(1)} = (m-1)! = n!$.

Proposition 6 *For* $\underline{k=2}$ *we get*

$$\beta_n^{(2)} = 1^2 \cdot 3^2 \cdot \ldots \cdot (2n-1)^2 = \left(\frac{(2n)!}{2^n n!}\right)^2.$$

Proof. A 3-brush with $n+1$ internal nodes (and $2(n+1)+1$ leaves) decomposes into three subtrees, where T_0 is just a leaf, T_1 has some i internal nodes and $2i+1$ leaves (for some $0 \le i \le n$), and T_2 has $n-i$ internal nodes and $2(n-i)+1$ leaves: see the figure below.

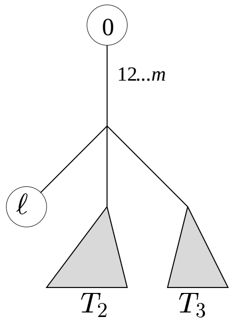
To determine one particular such tree, we first choose i; then there are $\binom{2(n+1)}{2(n-i)}$ choices for the leaf-labels of T_3, which must include the largest label m, and then there are $2i+1$ choices for the label of T_1 (which can be any but the largest among the remaining labels). Once the label sets are chosen, one has $\beta_i^{(2)}$ choices to determine T_2 and $\beta_{n-i}^{(2)}$ choices to determine T_3. This yields the recursion

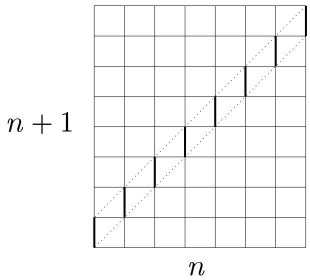
$$\beta_{n+1}^{(2)} = \sum_{i=0}^{n} (2i+1)\binom{2(n+1)}{2(i+1)} \beta_i^{(2)} \beta_{n-i}^{(2)} = \frac{(2(n+1))!}{2} \sum_{i=0}^{n} \frac{1}{i+1} \frac{\beta_i^{(2)}}{(2i)!} \frac{\beta_{n-i}^{(2)}}{(2(n-i))!}$$

for $n \ge 0$, with $\beta_0^{(2)} = 1$. Using the substitution $G_p = \frac{2^{2p}}{(2p)!}\beta_p^{(2)}$ resp. $\beta_p^{(2)} = \frac{(2p)!}{2^{2p}} G_p$, this reduces to

$$\tfrac{1}{2} G_{n+1} = \sum_{i=0}^{n} \frac{1}{i+1} G_i G_{n-i}$$

for $n \geq 0$, with $G_0 = 1$. To solve this, we note that $G_p = \binom{2p}{p}$ fits the recursion.





Namely, the number of monotone lattice paths in an $n \times (n+1)$ grid is $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{2}$. By counting the paths at the first edge where they cross the diagonal (at $x_1 = i$), we get

$$\tfrac{1}{2}\binom{2n}{2} \quad = \quad \sum_{i=0}^{n} \frac{1}{i+1}\binom{2i}{i}\binom{2(n-i)}{n-i},$$

using that the number of subdiagonal lattice paths in an $(i \times i)$-square is the Catalan number $C_i = \frac{1}{i+1}\binom{2i}{i}$. □

For small n, we analogously get $\beta_0^{(k)} = \beta_1^{(k)} = 1$ and

$$\beta_2^{(k)} = \binom{2k+1}{k} - 1.$$

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