1 Wissenschaftlicher Übersichtsartikel

Polytopes and Optimization: Recent Progress and Some Challenges[1]

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This is a discussion of four very active and important areas of research on the (combinatorial) theory of (convex) polytopes related to Linear Programming and to Combinatorial Optimization. I try to give a an account of recent progress, together with a selection of six "challenge" problems that I hope to see solved soon:

"Bad" linear programs and some combinatorial problems they pose.

Challenge: Is there a polynomial (in n and d) upper bound for the expected number of steps of the RANDOM EDGE simplex algorithm?

Challenge: Is the expected running time of the RANDOM EDGE simplex algorithm on the Klee-Minty cubes really quadratic?

"Worst" linear programs — extremal problems motivated by linear programming.

Challenge: The "Monotone Upper Bound Problem": What is the maximal number of vertices of a monotone path on a d-dimensional polytope with n facets?

0/1-polytopes and their combinatorial structure.

Challenge: The "0/1 Upper Bound Problem": Is the maximal number of facets of 0/1-polytopes bounded by an exponential function in the dimension?

Universality Theorems for polytopes of constant dimension.

Challenge: Can all 3-dimensional polytopes with m vertices be realized with coordinates of a size that is bounded by a polynomial in m?

Challenge: Provide a Universality Theorem for simplicial 4-dimensional polytopes.

In this area of research we have a wonderful mix of Optimization (the simplex algorithm for linear programming, structure of 0/1-polytopes for combinatorial optimization), Geometry (of convex polytopes) and Combinatorics (graphs, enumeration) that poses enough challenges for the future...

[1] A large part of this article is adapted, with permission, from the author's paper "Recent Progress on Polytopes," in: Proc. "Discrete and Computational Geometry: Ten Years Later," Mt. Holyoke, July 1996 (B. Chazelle, J.E. Goodman, R. Pollack, eds.), *Contemporary Mathematics*, Amer. Math. Soc., to appear.

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Disclaimer. This discussion is (solely) concerned with the **combinatorial** theory of **convex** polytopes, and with **recent** progress. It is a **personal selection** of topics, problems and directions that I consider to be interesting, partly because of their relations to and origins in questions that come from Linear Programming and Combinatorial Optimization. It is meant to be very informal, and cannot provide more than a sketch that hopefully makes you ask for more details and look at the references. In particular, look out for:

- The new edition of the "classic," Grünbaum [14],
- the *updates and more* offered for [27] on the Web,
- Ewald's new (1996) book [9], and
- Richter-Gebert's very recent (1997) book [24].

Notation. In the following, $P \subseteq \mathbb{R}^d$ always is a convex polytope of dimension d (a *d-polytope*) with m vertices and n facets.

1 "Bad" Linear Programs: The Klee-Minty cubes

A *linear program* is the task to find, with respect to a linear height function $c^t x$, a highest vertex of the set $P \subseteq \mathbb{R}^d$ of solutions of a set of n linear inequalities. We use here a very geometric setting of linear programming and the simplex algorithm (as in [27, Sect. 3.2]). That is, with the usual reductions we may assume without loss of generality that

- P is a bounded d-dimensional polytope,
- P has n facets (all inequalities are "facet-defining"),
- P is simple ("primal non-degeneracy"),
- P has no horizontal edges ("dual non-degeneracy"); in particular, the minimal and the maximal ("optimal") vertex with respect to $c^t x$ are unique, and
- the minimal vertex of P with respect to $c^t x$ is known (so "Phase I is done").

Our version of the simplex algorithm starts at the minimal vertex, and a *pivot rule* (which has only "local" information) chooses a path consisting of edges of P along which the objective function $c^t x$ increases strictly (a *monotone path*) until the optimal vertex is reached. At the core of linear programming theory we find the following two questions:

- Is there always a short (in terms of number of edges) path to the optimal vertex?
  (A very strong version of the Hirsch conjecture [27, p.87] would need such a path that has at most $n - d$ edges.)
- Can a simplex algorithm find one?
  (Is there a pivot rule for which the number of steps is bounded by a polynomial function of n and d? This would provide a strongly polynomial algorithm for linear programming!)

Unfortunately, virtually every known *deterministic* pivot rule has been shown to be exponential in the worst case on "deformed product programs": see Amenta & Ziegler [1].

However, it seems that this has been proved for *none* of the natural randomized pivot rules. The simplest one (to state) is the RANDOM EDGE rule: at every vertex choose, with equal probability, one of the increasing edges that leave the vertex. Easier to analyze seems to be the RANDOM FACET rule: if the increasing edge is not unique, choose randomly one of the facets that contains the current vertex, restrict the program to it, and solve recursively — for this rule one can, at least, establish sub-exponential upper bounds [17] [21].

Challenge 1 *Is the expected number of steps of the* RANDOM EDGE *simplex algorithm bounded by a polynomial function of n and d?*

Looking for especially "bad" linear programs, one is first led to the classical examples of linear programming theory, starting with the Klee-Minty cubes, the first and most important "deformed product programs." The d-dimensional Klee-Minty cube is given, for some ε with $0 < \varepsilon < \frac{1}{2}$, by

$$\max x_d : \quad \begin{array}{rcccl} 0 & \leq & x_1 & \leq & 1 \\ \varepsilon x_{j-1} & \leq & x_j & \leq & 1 - \varepsilon x_{j-1} \text{ for } 2 \leq j \leq d \end{array}$$

To analyze this, one has the following *Klee-Minty game* KM_n as a perfect combinatorial model. We start with a string of d 0s (corresponding to the vertex at the origin). Then for each step, one selects one 0 in the string, and flips this 0 *together with all the bits to its right.* (Here a *flip* changes a 0 into a 1, and a 1 into a 0.) Thus, for $d = 10$, we might get a sequence whose first six steps proceed as follows:

000000**0**000
000**0**001111
0011110**0**00
0**0**11110111
01000**0**1000
01**0**0010111
0111101000
....

The game stops when one reaches the string 1111...11 that does not have a 0. We know

- the ways to play the game KM_d are in bijection with the simplex paths on the d-dimensional Klee-Minty cube,
- the shortest game, choosing the leftmost 0, ends the game after only one step,
- the longest game, obtained if one always chooses the rightmost 0, takes $2^d - 1$ steps, and traces the "Klee-Minty path" through all the 2^d vertices of the d-dimensional Klee-Minty cube,
- the average length ℓ_d of the paths, all paths taken with equal weight, is exponential [10] (Bousquet-Mélou [6] just announced that $\ell_d \geq c2^d$ for some $c > 0$!), and
- choosing for each step one of the 0s *at random* (with equal probability, and independently) represents the behavior of the RANDOM EDGE simplex algorithm on the Klee-Minty d-cube.

For example, the game above was obtained by taking the 7th out of ten available (and equally likely) 0s for the first step, the 3rd out of six available 0s in the second step, the 4th out of six 0s in the third step, etc.

One can show that the expected number of steps is less than $.27d^2$ for large d from any starting vertex (an upper bound of $\binom{d+1}{2}$ is very easy to see), while it is more than d when starting from the zero string/vertex. That leaves a gap: what is the expected number of steps the RANDOM EDGE rule on the Klee-Minty game KM_d, if one starts with the zero string? We don't know! However, Gärtner, Henk & Ziegler [11, 10] established that the expected number of steps is at least $\frac{d^2}{4(\ln(d)-1)}$ *when starting at a random starting vertex.*

Challenge 2 *Is the expected number of steps of the* RANDOM EDGE *algorithm on the Klee-Minty cubes quadratic? That is, is there a constant $c > 0$ such that the expected number of steps of the Klee-Minty game* KM_d*, started at the zero string, and selecting a random* 0 *for each step, is at least cd^2?*

2 "Worst" Linear Programs: Upper Bound Problems

But what are the "worst" linear programs for the simplex algorithm? This question leads one, for each d and n, to consider the following hierarchy of *geometric* extremal problems:

$f_1(d,n)$: the maximal number of vertices of a 2-dim. projection	$\leq$	$f_2(d,n)$: the maximal number of vertices on a monotone path	$\leq$	$f_3(d,n)$: the maximal number of vertices

This is a very natural hierarchy: $f_1(d,n)$ is the largest number of steps for the simplex algorithm with the Gass-Saaty/Borgwardt shadow vertex rule [5] [18], $f_2(d,n)$ is the largest number of steps for the simplex algorithm with the most stupid choice of pivots, and $f_3(d,n)$ is a geometric upper bound that is known, $f_3(d,n) = \binom{n-\lceil d/2\rceil}{\lfloor d/2\rfloor} + \binom{n-\lceil (d+1)/2\rceil}{\lfloor (d-1)/2\rfloor}$, by the Upper Bound Theorem. (There is a very similar hierarchy for 0/1-polytopes; see Section 3 and [20].)

It is not at all clear whether the hierarchy collapses, that is, whether $f_1(d,n) = f_2(d,n) = f_3(d,n)$ holds for all n and d. This is true for $d \leq 3$. Also the three functions grow similarly fast — like polynomials of degree $\lfloor \frac{d}{2} \rfloor$ for constant d, and like exponential functions for constant $\frac{n}{d}$. However, for $d = 4$ we have indications for a gap: here we know that (among others) the polars of cyclic polytopes achieve the maximal numbers of $f_3(4,n) = \frac{n(n-3)}{2}$ vertices, they do not achieve the maximal number of vertices in a 2-dimensional shadow: any shadow can have at most $3n$ vertices [1].

Challenge 3 *Determine $f_2(d,n)$: what is the maximal number of vertices of a monotone path on a d-dimensional polytope with n facets?*

Let's look at the first interesting case, in dimension $d = 4$, with $n = 8$ facets. The 4-dimensional cube has these parameters, with $2^4 = 16$ vertices. The 4-dimensional Klee-Minty cube proves that, indeed, 16 vertices of a 4-dimensional polytope with 8 facets can lie on an increasing path. A further deformation of the Klee-Minty cubes, due to Murty [22] and Goldfarb [12] [1] even shows that all these 16 vertices of a 4-cube can appear as the vertices of a 2-dimensional projection.

However, the Upper Bound Theorem says that a 4-polytope with 8 facets can have as many as 20 vertices: take, for example, any "polar of a cyclic polytope" — a polytope denoted $C_4(8)^\Delta$ in [27]. Such a polytope can, for example, be written down explicitly by using facet coordinates on the Carathéodory curve, as

$$C_4(8)^\Delta := \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \;\middle|\; \begin{array}{r} \cos(\frac{k\pi}{2})x_1 + \sin(\frac{k\pi}{2})y_1 + \cos(2\frac{k\pi}{2})x_2 + \sin(2\frac{k\pi}{2})y_2 \le 1 \\ \text{for} \quad k \in \{0, 1, 2, \ldots, 6, 7\} \end{array} \right\}.$$

(If you don't like irrationals in your linear programs, and you shouldn't, then there is no problem in rounding $\frac{1}{2}\sqrt{2}$ to $\frac{7}{10}$, etc.) The problem is that we do not know how to choose a linear objective function such that we get a strictly monotone path that reaches more than 16 vertices. In this range of parameters we might even be able to find a 2-dimensional shadow of this polar-of-cyclic polytope that has more than 16 vertices, but we haven't, yet.

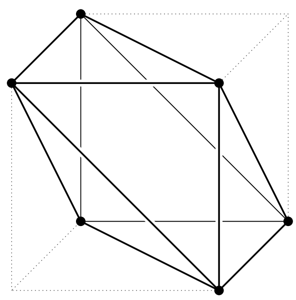
Putting all this together we get that

$$16 \le f_1(4, 8) \;\le\; f_2(4, 8) \;\le\; f_3(d, n) = 20,$$

so the gap is evident. (Thus the gap appears well within the range of computer experimentation!)

3 Extremal Properties of 0/1-Polytopes

A 0/1-*polytope* is a polytope of the form $P = \operatorname{conv}(V)$, where V is a set of 0/1-vectors, $V \subseteq \{0, 1\}^d$. Every $v \in V$ is a vertex of $P = \operatorname{conv}(V)$, and thus P is what one calls a *subpolytope* of the usual unit cube, $\operatorname{conv}(\{0, 1\}^d)$.



As an example, our figure shows an octahedron that arises as a 0/1-polytope in the unit 3-cube. However, in the study of 0/1-polytopes one should not rely too much on low-dimensional geometric intuition: things only become interesting (and complicated) in high dimensions.

About "general" 0/1-polytopes not much is known. However, there is a lot of details known about the "special" 0/1-polytopes that are associated with problems of Combinatorial Optimization such as the Travelling Salesman Problem (see Grötschel & Padberg [13]) and the Max-Cut Problem (see Deza & Laurent [8]). However, the division between "general" and "special" 0/1-polytopes is somewhat artificial, since a recent result of Billera & Sarangarajan [2] shows that *every* 0/1-polytope is (affinely equivalent to) a face of some asymmetric TSP-polytope.

The study of general 0/1-polytopes gets some of its motivation from core problems of Combinatorial Optimization such as TSP and Max-Cut: for example, one would like to know to what extent the special 0/1-polytopes are "typical." For that, it is interesting to describe the properties of "random 0/1-polytopes" as well as extremal properties, such as the following innocent(-looking) little "upper bound problem":

Challenge 4 *How fast does $f(d)$ grow with d, the maximal number of facets of a d-dimensional 0/1-polytope? In particular, is there a constant C such that $f(d) < C^d$ for all d?*

For small d, one can enumerate all possibilities, and this yields $f(d) = 2^d$ for $d \leq 4$ and $f(5) \geq 40$. The known general bounds

$$e^d < f(d) \leq (d-1)[(d-1)! + 2] \qquad \text{for all sufficiently large } d$$

derived in [20] leave a huge gap. Here the lower bound, recently improved to 3^d by R. Seidel (personal communication) is derived from direct sums of "centered" 0/1-polytopes. The upper bound uses a volume argument of I. Bárány.

How can one obtain 0/1-polytopes with "many" facets? It may well be that "random" 0/1-polytopes — with the right number of vertices — have many facets, but that may be hard to verify.

Perhaps one has more luck with very explicit constructions, such as cut polytopes? $\mathrm{CUT}(k) \subseteq \mathbb{R}^{\binom{k}{2}}$ is the convex hull of the incidence vectors of all cuts of a complete undirected graph K_k. After all, for small k the cut polytopes $\mathrm{CUT}(k)$ have "many" facets: $\mathrm{CUT}(7)$ has dimension $d = \binom{7}{2} = 21$ and $(1.743)^d$ facets, $\mathrm{CUT}(8)$ has dimension $d = \binom{8}{2} = 28$ and at least $(1.985)^d$ facets, while $\mathrm{CUT}(9)$ is currently outside the range of computation; See Christof's library of small polytopes SMAPO [7].

4 Universality

Realization spaces of point configurations (matroids resp. oriented matroids) as well as of polytopes appear in various places throughout Mathematics (under different names: moduli spaces, configuration spaces, etc.). In Optimization they appear, for example, in the stability analysis for linear and non-linear optimization, see e.g. Günzel, Hirabayashi & Jongen [16].

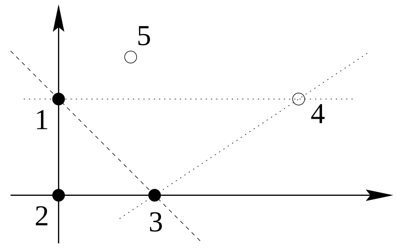
The *realization space* $\mathcal{R}(P)$ of P is the space of all matrices that coordinatize a polytope that is combinatorially equivalent to P, modulo affine equivalence. This space can be presented as a space of all matrices that satisfy a certain set of "determinantal" equations and strict inequalities, as in our example.

Example 4.1 (Pentagon) *We will construct $\mathcal{R}(C_2(5))$, where $C_2(5) \subseteq \mathbb{R}^2$ is a convex pentagon with vertices labelled $1,2,3,4,5$ in counter-clockwise order. The affine equivalence allows one to fix the first three vertices, for example as $\binom{x_1}{y_1} := \binom{0}{1}$, $\binom{x_2}{y_2} := \binom{0}{0}$, and $\binom{x_3}{y_3} := \binom{1}{0}$. With this we can write down the realization space of this pentagon as*

$$\mathcal{R}(C_2(5)) = \left\{ \begin{pmatrix} 0 & 0 & 1 & x_4 & x_5 \\ 1 & 0 & 0 & y_4 & y_5 \end{pmatrix} : \begin{array}{cc} x_4 > 0, & x_5 > 0, \\ y_4 > 0, & x_4 y_5 - x_5 y_4 + x_5 - x_4 > 0, \\ x_4 + y_4 > 0, & x_4 y_5 - x_5 y_4 + y_4 - y_5 > 0 \end{array} \right\} \subseteq \mathbb{R}^4.$$

Equivalently, one could fix the affine basis by explicitly fixing x_1, x_2, x_3 and y_1, y_2, y_3 and then using the conditions

$$\det \begin{pmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix} > 0 \quad \textit{for all} \ \ i < j < k.$$



In our figure, assume that the points $1,2,3$ have been fixed. The coordinate axes together with the dashed line bound an open polyhedron, which is the set of possible positions for the point 4. After that point has been fixed, the y-axis together with the two dotted lines bound the possible positions for 5, which again is an open polyhedron. From this one can see that $\mathcal{R}(C_2(5))$ is a 4-dimensional open semialgebraic set that has the topological type of an open 4-ball. Also, we can inductively construct rational coordinates (with "small" denominators and numerators) for the vertices of a pentagon. Similarly, for all $m \geq 3$ we get that $\mathcal{R}(C_2(m))$ is a topological $2(m-3)$-ball with "small rational points."

Steinitz's fundamental and classical theorem of 1922 [26, 24] states that the realization spaces of 3-polytopes are as "nice" as in the 2-dimensional case: they have the topological type of $(e-6)$-dimensional open balls (where $e(P)$ denotes the number of edges).

On the algebraic side, one can derive (not from Steinitz' proof!) that there is a singly-exponential bound for the vertex coordinates of 3-polytopes [23]. For example, all combinatorial types of rational 3-polytopes that have a triangle face can be realized with their m vertices placed in $\{0,1,2,\ldots,43^m\}^3$, as shown by Richter-Gebert [24]. The lower bound for the coordinate size needed is embarassingly low: of order $\Omega(m^{3/2})$, which is what one needs for a pyramid over an $(m-1)$-gon. This leads us to our first challenge problem.

Challenge 5 *Can all the combinatorial types of 3-dimensional polytopes be realized with integral coordinates whose size is bounded by a polynomial in the number of vertices?*

The situation in dimensions $d \geq 4$ is radically different from the 3-dimensional case. Mnëv's Universality Theorem (1988) implies that for polytopes with $d+4$ vertices the realization spaces are "universal." However, this does not answer the question for the situation in any fixed dimension.

I will not give a definition of "universal" here — this is usually expanded into "can be stably equivalent to an arbitrary primary semialgebraic set defined over $\mathbb{Z}$." Thus "universal" implies "(nearly) arbitrarily bad," in at least three respects:

- in terms of topology: all homotopy types of finite simplicial complexes do occur,
- in terms of singularities: all types of singularities of primary semialgebraic sets (defined over $\mathbb{Z}$) come up, and
- in terms of arithmetic properties: rational coordinates do not always exist, and no finite-dimensional extension of $\mathbb{Q}$ suffices to find coordinates for all combinatorial types.

I want to emphasize here that the "right" (in many respects) definition of *stably equivalent* and hence of *universal* is very important and original work by Richter-Gebert [24, 25]; all the previously used notions are either too strong to yield correct theorems, or they are too weak to derive all the three types of consequences, or they are unnatural or too difficult to handle.

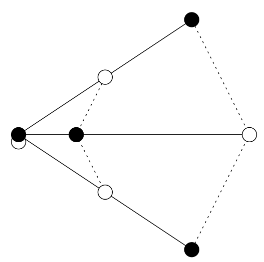
In 1994, Richter-Gebert completed a result that was long sought-after: a Universality Theorem for polytopes of some fixed dimension. (See also Günzel [15].)

Theorem 4.2 [Richter-Gebert [24]] *The realization spaces of* 4*-dimensional polytopes are universal.*

Challenge 6 *Is there a Universality Theorem for the realization spaces of* simplicial 4*-polytopes?*

Richter-Gebert's method cannot be applied here, since it depends on incidence theorems in the 2-skeleton, where simplicial polytopes don't have any. One can, however, expect that the realization spaces of simplicial 4-polytopes are not nice in general: on the one hand, the realization spaces of simplicial d-polytopes with $d+4$ vertices are complicated (in the topological sense) [4, Sect. 6.2]; and furthermore (exactly) one "non-trivial" example of a simplicial 4-polytope is known, the BEK polytope [3] [4, Sect. 6.2] with a disconnected realization space.

Richter-Gebert's work started with the analysis of some other very "small" and special 4-polytope. The crucial building block for his constructions turned out to be the 4-dimensional polytope P with 8 facets and 10 vertices whose polar is given by this affine Gale diagram:



The polytope P has a hexagon 2-face whose vertices necessarily (in every realization!) lie on an ellipse, thus answering (in the negative) a problem posed in [27, Problem 5.11*]. See Richter-Gebert [24] and Richter-Gebert & Ziegler [25] for a primal picture (Schlegel diagram) of this polytope. However, our little picture here may illustrate the power of the theory of Gale diagrams (most of it due to Perles [14], see [27, Lecture 6]): it allows one to analyze this interesting 4-dimensional polytope P in terms of a really simple 2-dimensional picture!

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