Recent Progress on Polytopes

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This is a discussion of five very active and important areas of research on the (combinatorial) theory of (convex) polytopes, with reports about recent progress, and a selection of seven "challenge" problems that I hope to see solved soon:

Universality Theorems for polytopes of constant dimension: see Richter-Gebert's work!
Challenge: Can all 3-dimensional polytopes be realized with coordinates of polynomial size?
Challenge: Provide a Universality Theorem for simplicial 4-dimensional polytopes.

Triangulations and subdivisions of polytopes.
Challenge: Decide whether all triangulations on a fixed point set in general position can be connected by bistellar flips.

0/1-polytopes and their combinatorial structure.
Challenge: The "0/1 Upper Bound Problem": Is the maximal number of facets of 0/1-polytopes bounded by an exponential function in the dimension?

Neighborly polytopes Explicit constructions and extremal properties.
Challenge: Is every polytope a quotient of a neighborly polytope?

Monotone paths and the simplex algorithm for linear programming.
Challenge: The "Monotone Upper Bound Problem": What is the maximal number of vertices of a monotone path on a d-dimensional polytope with n facets?
Challenge: Is there a polynomial upper bound for the running time of the RANDOM-EDGE simplex algorithm?

Disclaimer. This discussion is (solely) concerned with the **combinatorial** theory of **convex** polytopes — **recent** progress, and it is a **personal selection** of topics, problems and directions that I consider to be interesting. It is meant to be very informal, and cannot provide more than a sketch that hopefully makes you ask for more details and look at some of the references. Background material is in [16] and in [35]. Also watch for:

- The new edition of the "classic," Grünbaum [16],
- the *updates and more* offered for [35] on the Web,
- Ewald's new (1996) book [11], and
- Richter-Gebert's very recent (1997) book [27].

Notation. In the following our notation will be that $P \subseteq \mathbb{R}^d$ is a convex polytope of dimension d (a *d-polytope*) with m vertices and n facets.

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1 Universality.

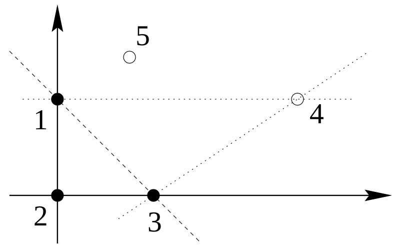
The *realization space* $\mathcal{R}(P)$ of P is the space of all matrices that coordinatize a polytope that is combinatorially equivalent to P, modulo affine equivalence. This space can be given as a space of all matrices that satisfy a certain set of "determinantal" equations and strict inequalities, as in our example.

Example 1.1 (Pentagon) *We will construct $\mathcal{R}(C_2(5))$, where $C_2(5) \subseteq \mathbb{R}^2$ is a convex pentagon with vertices labelled $1,2,3,4,5$ in counter-clockwise order. The affine equivalence allows one to fix the first three vertices, for example as $\binom{x_1}{y_1} = \binom{0}{1}$, $\binom{x_2}{y_2} = \binom{0}{0}$, and $\binom{x_3}{y_3} = \binom{1}{0}$. With this we can write down the realization space of a pentagon as*

$$\mathcal{R}(C_2(5)) \quad = \quad \left\{ \begin{pmatrix} 0 & 0 & 1 & x_4 & x_5 \\ 1 & 0 & 0 & y_4 & y_5 \end{pmatrix} : \begin{array}{cc} x_4 > 0, & x_5 > 0, \\ y_4 > 0, & x_4y_5 - x_5y_4 + x_5 - x_4 > 0, \\ x_4 + y_4 > 0, & x_4y_5 - x_5y_4 + y_4 - y_5 > 0 \end{array} \right\} \quad \subseteq \quad \mathbb{R}^4.$$

Equivalently, one could fix the affine basis by explicitly fixing x_1, x_2, x_3 and y_1, y_2, y_3 and then using the conditions

$$\det \begin{pmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix} \; > \; 0 \quad \text{for all} \quad i < j < k.$$



In our figure, assume that the points $1,2,3$ have been fixed. The coordinate axes together with the dashed line bound an open polyhedron, which is the set of possible positions for the point 4. After that point has been fixed, the y-axis together with the two dotted lines bound the possible positions for 5, which again is an open polyhedron. From this one can see that $\mathcal{R}(C_2(5))$ is a 4-dimensional open semialgebraic set that has the topological type of an open 4-ball. Also, we can inductively construct rational coordinates (with "small" denominators and numerators) for the vertices of a pentagon. Similarly, for all $m \geq 3$ we get that $\mathcal{R}(C_2(m))$ is a topological $2(m-3)$-ball with "small rational points."

Steinitz's classical theorem of 1922 [31, 27] states that the realization spaces of 3-polytopes are as "nice" as in the 2-dimensional case: they have the topological type of $(e-6)$-dimensional open balls (where $e(P)$ denotes the number of edges).

On the algebraic side, one can derive (not from Steinitz' proof!) that there is a singly-exponential bound for the vertex coordinates of 3-polytopes [25]. For example, all combinatorial types of rational 3-polytopes that have a triangle face can be realized with their m vertices placed in $\{0, 1, 2, \ldots, 43^m\}^3$, as shown by Richter-Gebert [27]. The lower bound for the coordinate size needed is embarassingly low: of order $\Omega(m^{3/2})$, which is what one needs for a pyramid over an $(m-1)$-gon. This leads us to our first challenge problem.

Challenge 1 *Can all 3-polytopes be realized with integral coordinates whose size is bounded by a polynomial in the number of vertices?*

The situation in dimensions $d \geq 4$ is radically different. From Mnëv's Universality Theorem (1988) one can derive that for polytopes with $d+4$ vertices the realization spaces are "universal." However, this does not answer the question for the situation in any fixed dimension.

I will not give a definition of "universal" here — this is usually expanded into "can be stably equivalent to an arbitrary primary semialgebraic set defined over $\mathbb{Z}$." Thus "universal" implies "(nearly) arbitrarily bad," in at least three respects:

- in terms of topology: all homotopy types of finite simplicial complexes do occur,
- in terms of singularities: all types of singularities of primary semialgebraic sets (defined over $\mathbb{Z}$) come up, and
- in terms of arithmetic properties: rational coordinates do not always exist, and no finite-dimensional extension of $\mathbb{Q}$ suffices to find coordinates for all combinatorial types.

I want to emphasize here that the "right" (in many respects) definition of *stably equivalent* and hence of *universal* is very important and original work by Richter-Gebert [27, 28]; all the previously used notions are either too strong to yield correct theorems, or they are too weak to derive all the three types of consequences, or they are unnatural or too difficult to handle.

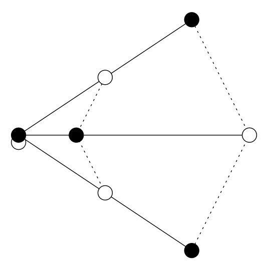
In 1994, Richter-Gebert completed a result that was long sought-after: a Universality Theorem for polytopes of some fixed dimension. (See also Günzel [18].)

Theorem 1.2 (Richter-Gebert [27]) *The realization spaces of* 4*-dimensional polytopes are universal.*

Challenge 2 *Is there a Universality Theorem for the realization spaces of* simplicial 4*-polytopes?*

Richter-Gebert's method cannot be applied here, since it depends on incidence theorems in the 2-skeleton, where simplicial polytopes don't have any. One can, however, expect that the realization spaces of simplicial 4-polytopes are not nice in general: on the one hand, the realization spaces of simplicial d-polytopes with $d+4$ vertices are complicated (in the topological sense) [4, Sect. 6.2]; and furthermore (exactly) one "non-trivial" example of a simplicial 4-polytope is known, the BEK polytope [3] [4, Sect. 6.2] with a disconnected realization space.

Richter-Gebert's work started with the analysis of some other very "small" and special 4-polytope. The crucial building block for his constructions turned out to be the 4-dimensional polytope X^* with 8 facets and 12 vertices whose polar is given by this affine Gale diagram:



The polytope X^* has a hexagon 2-face whose vertices satisfy a projective condition: two "opposite" edges and the corresponding diagonal intersect in one point (or are parallel), thus answering (in the negative) a problem posed in [35, Problem 5.11*]. See Richter-Gebert [27, p. 91] for a primal picture (Schlegel diagram) of this polytope. However, our little picture here may illustrate the power of the theory of Gale diagrams (most of it due to Perles [16], see [35, Lecture 6]): it allows one to analyze this interesting 4-dimensional polytope P in terms of a really simple 2-dimensional picture!

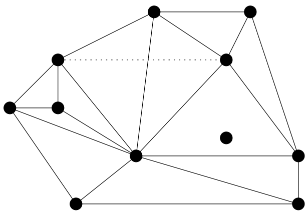
2 Triangulations and Bistellar Flips.

Fix a finite set $V \subseteq \mathbb{R}^d$ and let $P := \mathrm{conv}(V)$ be its convex hull. For most of this section we will assume that V is in general position (i.e., no $d+1$ of the points are contained in a hyperplane), in order to make the following definitions as simple as possible. A *triangulation* of V is a triangulation of P whose vertex set is contained in V. (A nice new survey on triangulations of polytopes is Lee [23].)

A *(bistellar) flip* in a triangulation is a certain type of local change: In the general position case this operation replaces a subcomplex that represents the d-dimensional "picture of" a front side of a $(d+1)$-dimensional simplex by its back side.

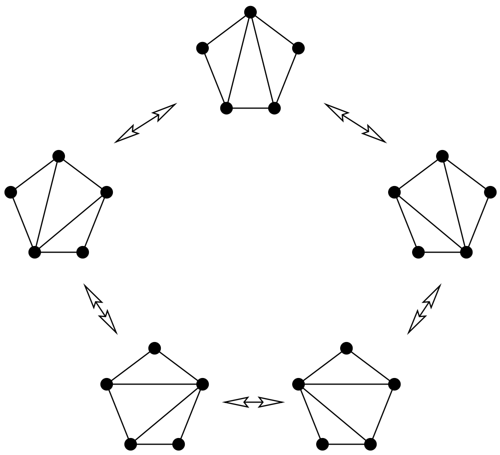
A triangulation is *regular* if it is a "correct" picture of the front side of a simplicial $(d+1)$-polytope. This is a global property that may be destroyed (or created) by a flip.

For example, our next figure represents a planar set V of 11 points, whose convex hull is a 7-gon. The straight lines indicate a triangulation that uses only 10 of the 11 points. One dotted line (that would replace the straight line it crosses) represents one possible flip that one could visualize as replacing the two front faces of a tetrahedron by its two back faces. Both triangulations of the figure (before and after the flip) are regular (Exercise!).



Triangulations are of tremendous importance for topics that range from the construction of splines (PDE, CAD, ...) to algebraic geometry (the resolutions of toric varieties). In view of this there is a surprising variety of very basic, very interesting and very open problems connected to the set of triangulations of a finite point set V.

To describe some of these, we consider the graph $G(V)$ whose vertex set is the (finite) set of all triangulations of V, and whose edges correspond to all possible bistellar flips. For example, if V is the vertex set of a pentagon, then the graph $G(V)$ that one obtains is a five-cycle, as one can see from our figure.

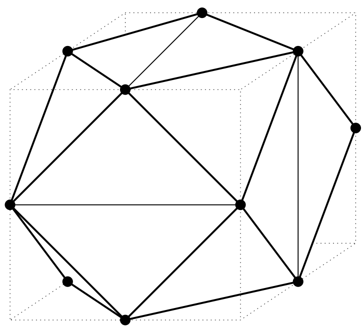


The induced subgraph of $G(V)$ that contains the *regular* triangulations of V and their bistellar flips will be denoted by $G_r(V)$.

Theorem 2.1 (Gel'fand-Kapranov-Zelevinsky [14]) *For every set $V \subseteq \mathbb{R}^d$ of $m > d$ points the graph of regular triangulations $G_r(V)$ is the graph of an $(m-d-1)$-dimensional polytope, the "secondary polytope" $\Sigma(V)$.*

Thus (by "Balinski's Theorem") the graph $G_r(V)$ is $(m-d-1)$-vertex-connected; in particular, it is connected and its minimum degree is at least $m-d-1$. The following configuration (found in 1996, see also [7]) indicates that the same is not true if we consider the complete graph $G(V)$.

Example 2.2 (de Loera, Santos & Urrutia [8]) *For $d = 3$ and $m = 13$, let V consist of the 12 midpoints of edges and of the center of a regular cube. Then P is a polytope (truncated cube) whose boundary may be triangulated by adding 6 additional, independent edges, as in our figure:*



(The same figure can be found in Coxeter [6, p.152]!) From this we obtain a triangulation of P for which every tetrahedron uses the center point of the original cube. (This is a triangulation with 13 vertices, $30 + 12 = 42$ edges, $20 + 30 = 50$ triangles and 20 tetrahedra.) It is easy to check that only $6 = m-d-4$ flips are possible. (Thus, in particular, the triangulation cannot be regular.) The point configuration can be perturbed to general position, maintaining this property. It can also be lifted to a $(d' = 4)$-configuration of $m' = 14$ points in convex position $\mathbb{R}^4$, where we still have only $6 = m' - d' - 4$ flips.

This was an example with "few" flips: but is there always at least one flip possible?

Challenge 3 *Given m points in $\mathbb{R}^d$ (in general position), is the graph of all triangulations connected by bistellar flips?*

This is known to be true for $d \leq 2$. It was proved for $m - d \leq 3$ by Lee [22], who showed that then all triangulations are regular. For all cyclic polytopes $C_d(m)$ it is true by very recent work of Rambau [26]; except for that, the problem is open. I expect a negative answer.

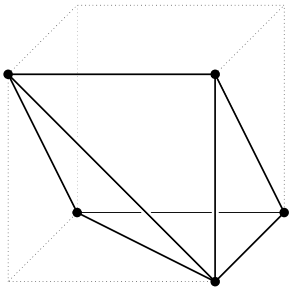
In the case of cyclic polytopes one can use the "higher Stasheff-Tamari posets" introduced by Edelman & Reiner [10], natural partial orders on the set of all triangulations of a cyclic polytope $C_d(m)$, by putting them into correspondence with triangulated hypersurfaces in $C_{d+1}(m)$:

$$\begin{array}{c} \text{hypersurfaces in } C_{d+1}(m) \\ \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \\ \text{triangulations of } C_d(m) \end{array}$$

Rambau's proof proceeds by establishing a sequence of simple properties of cyclic polytopes and their triangulations: interestingly enough, many of them are easily seen to be *false* in general. For example, a key step is to establish that for the $(d+1)$-simplices of $C_{d+1}(m)$, the "on top of" relation generates a partial order. (This is false for a perturbation of the "capped prism" of Lee [23, Sect. 3.3].) But the fact that proofs don't generalize doesn't show that the general conjecture is false...

3 Extremal Properties of 0/1-Polytopes.

A 0/1-*polytope* is a polytope of the form $P = \text{conv}(V)$, where V is a set of 0/1-vectors, $V \subseteq \{0,1\}^d$. Every $v \in V$ is a vertex of $P = \text{conv}(V)$, and thus P is what one calls a *subpolytope* of the usual unit cube, $\text{conv}(\{0,1\}^d)$.



As an example, our figure shows a square pyramid that arises as a 0/1-polytope in the unit 3-cube. However, in the study of 0/1-polytopes one should not rely too much on low-dimensional geometric intuition: things only become interesting (and complicated) in high dimensions.

About "general" 0/1-polytopes not much is known. However, there are many details known about the "special" 0/1-polytopes that are associated with problems of combinatorial optimization such as the Travelling Salesman Problem (see Grötschel & Padberg [15]) and Max-Cut Problems (see Deza & Laurent [9]). However, the division between "general" and "special" 0/1-polytopes is somewhat artificial, since a recent result of Billera & Sarangarajan [2] shows that *every* 0/1-polytope is (affinely equivalent to) a face of some asymmetric TSP-polytope.

The study of general 0/1-polytopes gets some of its motivation from core problems of combinatorial optimization such as TSP and Max-Cut: for example, one would like to know how "typical" the special 0/1-polytopes are. For that, it is interesting to describe the properties of "random 0/1-polytopes" as well as extremal properties, such as the following innocent(-looking) little "upper bound problem":

Challenge 4 *How fast does $f(d)$, the maximal number of facets of a d-dimensional* 0/1*-polytope grow with d? In particular, is there a constant C such that $f(d) < C^d$ for all d?*

For small d, one can enumerate all possibilities, and this yields $f(d) = 2^d$ for $d \leq 4$ and $f(5) \geq 40$. The best known general bounds (as of January 1997),

$$(3.26)^d < f(d) \leq \frac{6.4\, d!}{\sqrt{d}} \qquad \text{for all sufficiently large } d,$$

leave a huge gap. Here the lower bound is derived from direct sums of "centered" 0/1-polytopes, see [21]. The constant 3.26 is derived from a specific example of a 12-dimensional 0/1-polytope with 1489211 facets — see [20] for this and other current records in small dimensions. The upper bound, by G. Rote [29] uses estimates for the volume and the surface area.

How can one obtain 0/1-polytopes with "many" facets? It may well be that "random" 0/1-polytopes — with the right number of vertices — have many facets, but that may be hard to verify.

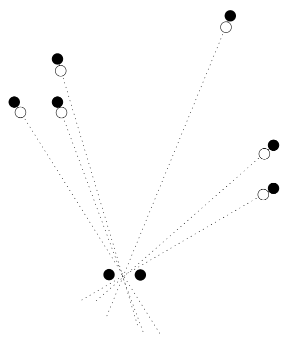
Perhaps one has more luck with very explicit constructions, such as cut polytopes? Here $\text{CUT}(k) \subseteq \mathbb{R}^{\binom{k}{2}}$ is the convex hull of the incidence vectors of all cuts of a complete undirected graph K_k. After all, for small k the cut polytopes $\text{CUT}(k)$ have "many" facets: CUT(7) has dimension $d = \binom{7}{2} = 21$ and $(1.743)^d$ facets, while CUT(8) has dimension $d = \binom{8}{2} = 28$ and at least $(1.985)^d$ facets, where CUT(9) is currently outside the range of computation; See Christof's library of small polytopes SMAPO [5].

4 Constructing Neighborly Polytopes.

Neighborly polytopes (4-polytopes such that any two vertices are adjacent; 6-polytopes such that every set of three vertices determines a triangle 2-face, etc.) are important. They have a prominent role in the theory of convex polytopes due to their extremal properties in terms of their numbers of faces, as given by the Upper Bound Theorem of McMullen [35, Thm. 8.23], which got a notable strengthening recently in Novik [24].

Neighborly polytopes exist; in fact, the cyclic polytopes $C_d(n)$ — our prime examples — are easy to write down in explicit coordinates (using the moment curve) and to analyze (using Vandermonde determinants). Also, we know that there are really many neighborly polytopes (Shemer [30]) and in some models of random polytopes it seems that even random polytopes are often neighborly. Gale wrote "the likelihood of getting a neighborly polytope increases rapidly with the dimension of the space" a long time ago [13, p. 262], but despite several attempts this has not been proven in any form.

In fact, it is not that easy to write down many neighborly polytopes. However, in recent work Kortenkamp [19] has provided a new method to construct neighborly polytopes with $m = d + 4$ vertices — they can be constructed and analyzed in terms of their 2-dimensional affine Gale diagrams. For this, take any finite configuration of m "black" points in the plane (no three on a line). From it we get a "balanced" configuration of $m + 1$ black points and $m - 1$ white points:



Here a configuration of points in general position is "balanced" if for every hyperplane spanned by some of them one has

$$\text{surplus of black points on one side} \quad = \quad \text{surplus of white points on the other side.}$$

This condition also characterizes the Gale diagrams of neighborly polytopes, and thus (with the usual gymnastics and formalism of Gale diagrams) our little figure proves the following theorem.

Theorem 4.1 (Kortenkamp [19]) *Every simplicial d-polytope with $m \leq d+4$ vertices is equivalent to a quotient (iterated vertex figure) of a neighborly $(2d+4)$-polytope with $2d+8$ vertices.*

To extend this result to arbitrary simplicial polytopes one "only" needs a construction that embeds every point configuration in general position in $\mathbb{R}^d$ into a "balanced" configuration.

Challenge 5 (Perles [16, 32]) *Is every simplicial d-polytope (with vertices in general position) a quotient (iterated vertex figure) of a neighborly polytope?*

Note, however, that some simple-sounding embedding problems are notoriously difficult, such as the problem of embedding any matroid of rank 3 into a finite projective plane (this is believed to be possible; see Wanner & Ziegler [34]), or the problem of embedding every line arrangement in the plane into a simplicial line arrangement (this is believed to be impossible in general; see Grünbaum [17, p. 9]).

5 Monotone Paths on Polytopes.

A *linear program* is the task to find, with respect to a linear height function $c^t x$, a highest vertex of the set $P \subseteq \mathbb{R}^d$ of solutions of a set of n linear inequalities. We deal here with a very geometric setting of linear programming and the simplex algorithm (as in [35, Sect. 3.2]). That is, with the usual reductions we may assume without loss of generality:

- P is a bounded d-dimensional polytope,
- P has n facets, all inequalities are facet-defining,
- P is simple ("primal non-degeneracy"),
- P has no horizontal edges ("dual non-degeneracy"), so that in particular the minimal and the maximal ("optimal") vertex with respect to $c^t x$ are unique, and
- the minimal vertex of P with respect to $c^t x$ is known ("Phase I is done").

Our version of the simplex algorithm starts at the minimal vertex, and a *pivot rule* (which has only "local" information) chooses a path consisting of edges that improve the objective function $c^t x$ (a *monotone path*) until the optimal vertex is reached. At the core of linear programming theory we find the following two questions:

- Is there always a short (in terms of number of edges) path to the optimal vertex? (A very strong version of the Hirsch conjecture [35, p.87] would need such a path that has at most $n - d$ edges), and
- Can a simplex algorithm find one? (Is there a pivot rule for which the number of steps is bounded by a polynomial function of n and d? This would provide a strongly polynomial algorithm for linear programming!)

Unfortunately, virtually every *deterministic* pivot rule has been shown to be exponential in the worst case on certain "deformed product programs": see Amenta & Ziegler [1]. However, it seems that this has been proved for *none* of the natural randomized pivot rules. The simplest one (to state) is the RANDOM-EDGE rule: at every vertex choose, with equal probability, one of the increasing edges that leave the vertex.

Challenge 6 *Is the expected number of steps of the RANDOM-EDGE simplex algorithm bounded by a polynomial function of n and d?*

Looking for especially "bad" linear programs, one is first led to the classical examples of linear programming theory, starting with the Klee-Minty cubes: Here RANDOM-EDGE is polynomial, with quadratic upper bound (which is easy to see) and with a nearly-quadratic lower bound (which takes some effort to show [12]).

But what are the "worst" linear programs for the simplex algorithm? This question leads one, for each d and n, to consider the following hierarchy of geometric extremal problems:

$f_1(d,n)$: the maximal number of vertices of a 2-dim. projection	$\leq$	$f_2(d,n)$: the maximal number of vertices on a monotone path	$\leq$	$f_3(d,n)$: the maximal number of vertices

This is a very natural hierarchy: $f_1(d,n)$ is the largest number of steps for the simplex algorithm with the Gass-Saaty/Borgwardt shadow vertex rule, $f_2(d,n)$ is the largest number of steps for the simplex algorithm with the most stupid choice of pivots, and $f_3(d,n)$ is a geometric upper bound that is known, by the Upper Bound Theorem. (There is a very similar hierarchy for 0/1-polytopes; see Section 3 and [21].)

It is not at all clear whether the hierarchy collapses, that is, whether $f_1(d,n) = f_2(d,n) = f_3(d,n)$ holds for all n and d. This is true for $d \leq 3$. Also the three functions grow similarly fast — like polynomials of degree $\lfloor \frac{d}{2} \rfloor$ for constant d, and like exponential functions for constant $\frac{n}{d}$. However, for $d = 4$ we have indications for a gap: here we know that polars of cyclic polytopes achieve the maximal numbers of $f_3(d,n)$ vertices, they do not achieve the maximal number of vertices in a 2-dimensional shadow [1].

Challenge 7 *Determine $f_2(d,n)$: what is the maximal number of vertices of a monotone path on a d-dimensional polytope with n facets?*

In this area of research we have a wonderful mix of optimization (LP: simplex algorithm), geometry (polytopes) and combinatorics (graphs, enumeration) that poses enough challenges for the future...

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