ORIENTED MATROIDS

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INTRODUCTION

The theory of *oriented matroids* provides a broad setting in which to model, describe, and analyze combinatorial properties of geometric configurations. Apparently totally different mathematical objects such as *point and vector configurations*, *arrangements of hyperplanes*, *convex polytopes*, *directed graphs*, and *linear programs* find a common generalization in the language of oriented matroids.

The oriented matroid of a finite set of points $\boldsymbol{P}$ extracts "relative position" and "orientation" information from the configuration; for example, it can be given by a list of signs that encodes the orientations of all the bases of $\boldsymbol{P}$. In the passage from a concrete point configuration to its oriented matroid metrical information is lost, but many structural properties of $\boldsymbol{P}$ have their counterparts at the — purely combinatorial — level of the oriented matroid.

We first introduce oriented matroids in the context of several models and motivations (Section 7.1). Then we present some equivalent axiomatizations (Section 7.2). Finally, we discuss concepts that play central roles in the theory of oriented matroids (Section 7.3), among them *duality*, *realizability*, the study of *simplicial cells*, and the treatment of *convexity*.

7.1 MODELS AND MOTIVATIONS

This section discusses geometric examples that are usually treated on the level of concrete coordinates, but where an "oriented matroid point of view" gives deeper insight. We also present these examples as standard models that provide intuition for the behaviour of general oriented matroids.

7.1.1 ORIENTED BASES OF VECTOR CONFIGURATIONS

GLOSSARY

Vector configuration: A matrix $X = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$, usually assumed to have full rank d.

Matroid of X: The pair $M_X = (E, \mathcal{B}_X)$, where $E := \{1, 2, \ldots, n\}$ and $\mathcal{B}_X$ is the set of all (column indices) of bases of X.

Matroid: A pair $M = (E, \mathcal{B})$, where E is a finite set, and $\mathcal{B} \subseteq 2^E$ is a nonempty collection of subsets of E (the *bases* of M) that satisfies the *Steinitz exchange axiom*: For all $B_1, B_2 \in \mathcal{B}$ and $e \in B_1 \backslash B_2$, there exists an $f \in B_2 \backslash B_1$ such that $(B_1 \backslash e) \cup f \in \mathcal{B}$.

Signs: Elements of the set $\{-, 0, +\}$, used as a shorthand for the corresponding elements of $\{-1, 0, +1\}$.

Chirotope of X: The map

$$\begin{array}{rrcl} \chi_X: & E^d & \to & \{-, 0, +\} \\ & (\lambda_1, \ldots, \lambda_d) & \mapsto & \operatorname{sign}(\det(x_{\lambda_1}, \ldots, x_{\lambda_d})). \end{array}$$

Ordinary (unoriented) *matroids*, as introduced in 1935 by Whitney (see Oxley [15]), can be considered as an abstraction of vector configurations in finite dimensional vector spaces over arbitrary fields. All the bases of a matroid M have the same cardinality d, which is called the *rank* of the matroid. Equivalently, we can identify M with the characteristic function of the bases $B_M: E^d \to \{0, 1\}$, where $B_M(\lambda) = 1$ if and only if $\{\lambda_1, \ldots, \lambda_d\} \in B$.

One can obtain examples of matroids as follows: take a finite set of vectors

$$X = \{x_1, x_2, \ldots, x_n\} \subseteq K^d$$

of rank d in an arbitrary vector space K^d and consider the set of bases of K^d formed by subsets of the points in X. In other words, the pair

$$M_X = (E, B_X) = \Big(\{1, \ldots, n\},\ \{\{\lambda_1, \ldots, \lambda_d\} \ :\ \det(x_{\lambda_1}, \ldots, x_{\lambda_d}) \neq 0\} \Big)$$

forms a matroid.

The basic information about the incidence structure of the points in X is contained in the underlying matroid M_X. However, the matroid alone presents only a weak model of a geometric configuration; for example, all configurations of n points ***in general position*** in the plane (i.e., no three points on a line) have the same matroid $M = U_{3,n}$: here no information beyond the dimension and size of the configuration, and the fact that it is in general position, is retained for the matroid.

In contrast to matroids, the theory of *oriented matroids* considers the structure of dependencies in vector spaces over *ordered* fields. Roughly speaking, an oriented matroid is a matroid where in addition every basis is equipped with an orientation. These oriented bases have to satisfy an oriented version of the Steinitz exchange axiom (to be described later). In other words, oriented matroids not only describe the incidence structure between the points of X and the hyperplanes spanned by points of X (this is the matroid information); they also encode the positions of the points relative to the hyperplanes: "Which points lie on the positive side of a hyperplane, which points lie on the negative side, and which lie on the hyperplane?" If $X \in (K^d)^n$ is a configuration of n points in a d-dimensional vector space K^d over an ordered field K, we can describe the corresponding oriented matroid χ_X by the function:

$$\begin{array}{rrcl} \chi_X: & E^d & \to & \{-, 0, +\} \\ & (\lambda_1, \ldots, \lambda_d) & \mapsto & \operatorname{sign}(\det(x_{\lambda_1}, \ldots, x_{\lambda_d})). \end{array}$$

This map χ_X is called the ***chirotope*** of X and is very closely related to the oriented matroid of X. It encodes much more information than the corresponding matroid, including information about the *topology* and the *convexity* of the underlying configurations.

7.1.2 ARRANGEMENTS OF POINTS

GLOSSARY

Affine point configuration: A matrix $(p_1, \ldots, p_n) \in (\mathbb{R}^{d-1})^n$, usually assumed to have full rank $d-1$.

Associated vector configuration: The matrix $X \in (\mathbb{R}^d)^n$ obtained from a point configuration by adding a row of ones. This corresponds to the embedding of the affine space $\mathbb{R}^{d-1}$ into the linear vector space $\mathbb{R}^d$ via $p \longmapsto x = \binom{p}{1}$.

Oriented matroid of a point configuration: The oriented matroid of the associated vector configuration.

Covector of a vector configuration X: Partition of $X = (x_1, \ldots, x_n)$ induced by a linear hyperplane, into points on the hyperplane, on the positive side, and on the negative side.

Oriented matroid of X: The collection $\mathcal{L} \subseteq \{-, 0, +\}^n$ of all covectors of X.

Let $X := (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ be an $n \times d$ matrix and let $E := \{1, \ldots, n\}$. We interpret the columns of X as n vectors in the d-dimensional real vector space $\mathbb{R}^d$. For a linear functional $y^T \in (\mathbb{R}^d)^*$ we set

$$C_X(y) = (\mathrm{sign}(y^T x_1), \ldots, \mathrm{sign}(y^T x_n)).$$

Such a sign vector is called a ***covector*** of X. We denote the collection of all covectors of X by

$$\mathcal{L}_X := \{C_X(y) \ : \ y \in \mathbb{R}^d\}.$$

The pair $\mathcal{M}_X = (E, \mathcal{L}_X)$ is called the ***oriented matroid*** of X. Here each sign vector $C_X(y) \in \mathcal{L}_X$ describes the positions of the vectors $x_1, \ldots, x_n$ relative to the linear hyperplane $H_y = \{x \in \mathbb{R}^d \ : \ y^T x = 0\}$: the sets

$$\begin{array}{rcl} C_X(y)^0 & := & \{e \in E \ : \ C_X(y)_e = 0\} \\ C_X(y)^+ & := & \{e \in E \ : \ C_X(y)_e > 0\} \\ C_X(y)^- & := & \{e \in E \ : \ C_X(y)_e < 0\} \end{array}$$

describe how H_y partitions the set of points X. Here $C_X(y)^0$ contains the points on H_y, while $C_X(y)^+$ and $C_X(y)^-$ contain the points on the positive and on the negative side of H_y, respectively. In particular, if $C_X(y)^- = \emptyset$, then all points not on H_y lie on the positive side of H_y. In other words, in this case H_y determines a face of the positive cone

$$\mathrm{pos}(x_1, \ldots, x_n) := \Big\{ \lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n \ : \ 0 \le \lambda_i \in \mathbb{R} \text{ for } 1 \le i \le n \Big\}$$

of all points of X. The face lattice of the cone $\mathrm{pos}(X)$ can be recovered from $\mathcal{L}_X$. It is simply the set $\mathcal{L}_X \cap \{+, 0\}^E$, partially ordered by the order induced from the relation "$0 < +$."

If in the configuration X we have $x_{i,d} = 1$ for all $1 \le i \le n$, then we can consider X as representing homogeneous coordinates of an *affine* point set X' in $\mathbb{R}^{d-1}$. Here the affine points correspond to the original points x_i after removal of the d-th coordinate. The face lattice of the convex polytope $\mathrm{conv}(X') \subseteq \mathbb{R}^{d-1}$ is then identical to the face lattice of $\mathrm{pos}(X)$. Hence, $\mathcal{M}_X$ can be used to recover the *convex hull* of X'.

Thus oriented matroids are generalizations of point configurations in linear or affine spaces. For general oriented matroids we weaken the assumption that the hyperplanes spanned by points of the configuration are really flat to the assumption that they only satisfy certain topological incidence properties. Nonetheless, this kind of picture is sometimes misleading since not all oriented matroids have this type of representation (compare the "Type II representations" of [3, Sect. 5.3]).

7.1.3 ARRANGEMENTS OF HYPERPLANES AND OF HYPERSPHERES

GLOSSARY

Hyperplane arrangement $\mathcal{H}$: Collection of (oriented) linear hyperplanes in $\mathbb{R}^d$, given by normal vectors $x_1, \ldots, x_n$.

Hypersphere arrangement induced by $\mathcal{H}$: Intersection of $\mathcal{H}$ with the unit sphere S^{d-1}.

Covectors of $\mathcal{H}$: Sign vectors of the cells in $\mathcal{H}$; equivalently, $\mathbf{0}$ together with the sign vectors of the cells in $\mathcal{H} \cap S^{d-1}$.

We obtain a different picture if we polarize the situation and consider *hyperplane arrangements* rather than arrangements of points. For a real matrix $X := (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ consider the system of hyperplanes $\mathcal{H}_X := (H_1, \ldots, H_n)$ with

$$H_i := \{y \in \mathbb{R}^d \ : \ y^T x_i = 0\}.$$

Each vector x_i induces an orientation on H_i by defining

$$H_i^+ := \{y \in \mathbb{R}^d \ : \ y^T x_i > 0\}$$

to be the ***positive side*** of H_i. We define H_i^- analogously to be the ***negative side*** of H_i. To avoid degenerate cases we assume that X contains at least one proper basis (i.e., the matrix X has rank d). The hyperplane arrangement $\mathcal{H}_X$ subdivides $\mathbb{R}^d$ into polyhedral cones. Without loss of information we can intersect with the unit sphere S^{d-1} and consider the sphere system

$$\mathcal{S}_X \ := \ (H_1 \cap S^{d-1}, \ldots, H_n \cap S^{d-1}) \ = \ \mathcal{H}_X \cap S^{d-1}.$$

Our assumption that X contains at least one proper basis translates to the fact that the intersection of all $H_1 \cap \ldots \cap H_n \cap S^{d-1}$ is empty. $\mathcal{H}_X$ induces a cell decomposition $\Gamma(\mathcal{S}_X)$ on S^{d-1}. Each face of $\Gamma(\mathcal{S}_X)$ corresponds to a sign vector in $\{-, 0, +\}^E$ that indicates the position of the cell with respect to the $(d-2)$-spheres $H_i \cap S^{d-1}$ (and therefore with respect to the hyperplanes H_i) of the arrangement. The list of all these sign vectors is exactly the set $\mathcal{L}_X$ of covectors of $\mathcal{H}_X$.

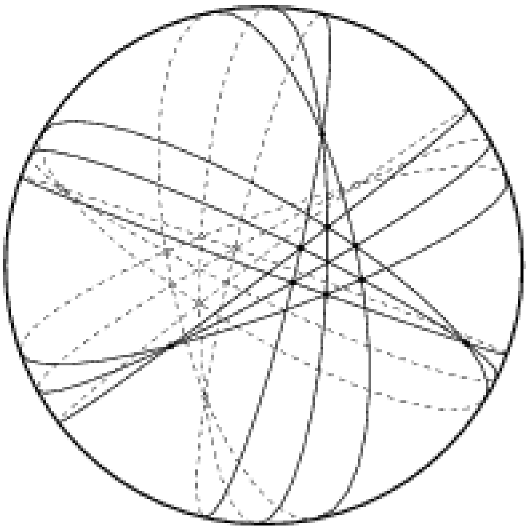


Figure 7.1
An arrangement of nine great circles on S^2. The arrangement corresponds to a Pappus configuration.

While the visualization of oriented matroids by sets of points in $\mathbb{R}^n$ does not fully generalize to the case of non-representable oriented matroids, the picture of hyperplane arrangements has a well-defined analogue that also covers all the non-realizable cases. We will see that as a consequence of the Topological Representation Theorem of Folkman & Lawrence (Section 7.2.4) every rank d oriented matroid can be represented as an arrangement of oriented *pseudospheres* (or pseudohyperplanes) embedded in the S^{d-1} (resp. in $\mathbb{R}^d$). Arrangements of pseudospheres are systems of topological $(d-2)$-spheres embedded in S^{d-1} that satisfy certain intersection properties that clearly hold in the case of "straight" arrangements.

7.1.4 ARRANGEMENTS OF PSEUDOLINES

GLOSSARY

Pseudoline: Simple closed curve p in the projective plane $\mathbb{RP}^2$ that is topologically equivalent to a line (i.e., there is a self-homeomorphism of $\mathbb{RP}^2$ mapping p to a straight line).

Arrangement of pseudolines: Collection of pseudolines $\mathcal{P} := (p_1, \ldots, p_n)$ in the projective plane, where any two of them intersect exactly once.

Simple arrangement: No three pseudolines meet in a common point. (Equivalently, the associated oriented matroid is ***uniform***.)

Equivalent arrangements: Arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$ that generate isomorphic cell decompositions of $\mathbb{RP}^2$. (In this case there exists a self-homeomorphism of $\mathbb{RP}^2$ mapping $\mathcal{P}_1$ to $\mathcal{P}_2$.)

Stretchable arrangement of pseudolines: An arrangement that is equivalent to an arrangement of projective lines.

An *arrangement of pseudolines* in the projective plane is a collection of pseudolines such that any two pseudolines intersect in exactly one point, where they cross. (See Grünbaum [10] and Richter [16].) We will always assume that $\mathcal{P}$ is ***essential***, i.e., that the intersection of all the pseudolines p_i is empty.

An arrangement of pseudolines behaves in many respects just like an arrangement of n lines in the projective plane. (In fact, there are only very few combinatorial theorems known that are true for straight arrangements, but not true in general for pseudoarrangements.)

Figure 7.2 shows a small example of a non-stretchable arrangement of pseudolines. (It is left as a challenging exercise to the reader to prove the non-stretchability.) Up to isomorphism this is the only simple non-stretchable arrangement of 9 pseudolines [16] [11]; every arrangement of 8 (or fewer) pseudolines is stretchable.

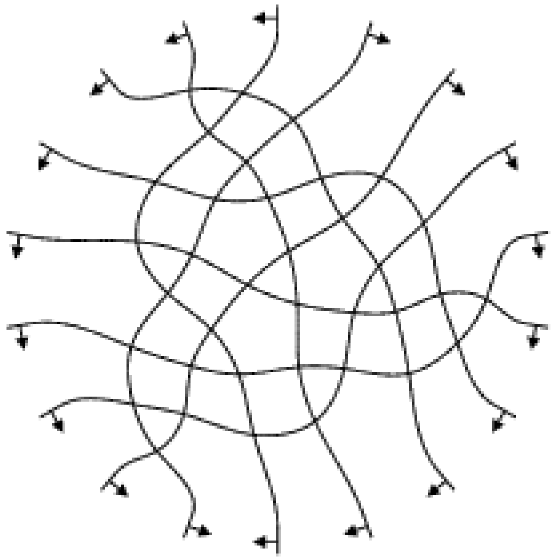


Figure 7.2
A non-stretchable arrangement of nine pseudolines. It was obtained by Ringel [18] *as a perturbation of the Pappus configuration.*

To associate with a projective arrangement $\mathcal{P}$ an oriented matroid we represent the projective plane (as customary) by the 2-sphere on which antipodal points are identified. With this every arrangement of pseudolines gives rise to an arrangement of *great pseudocircles* on S^2. For each great pseudocircle on S^2 we choose a positive side. Each cell induced by $\mathcal{P}$ on S^2 now corresponds to a unique sign vector. The collection of all these sign vectors again forms a set of covectors $\mathcal{L}_{\mathcal{P}}\backslash\mathbf{0}$ of an oriented matroid of rank 3. Conversely, as a special case of the Topological Representation Theorem, *every* oriented matroid of rank 3 has a representation by an *oriented* pseudoline arrangement.

In this way we can use pseudoline arrangements as a standard picture to represent rank 3 oriented matroids. The easiest picture is obtained when we restrict ourselves to the upper hemisphere of S^2 and assume w.l.o.g. that each pseudoline crosses the equator exactly once, and that the crossings are distinct (i.e., no intersection of the great pseudocircles lies on the equator). Then we can represent this upper hemisphere by an arrangement of mutually crossing, oriented affine pseudolines in the plane $\mathbb{R}^2$. (We did this implicitly while drawing Figure 7.2.)

By means of this equivalence, all problems concerning arrangements of pseudolines can be translated to the language of oriented matroids. For instance the problem of stretchability is equivalent to the realizability problem for oriented matroids.

7.2 AXIOMS AND REPRESENTATIONS

In this section we define oriented matroids formally. It is one of the main features of oriented matroid theory that the same object can be viewed under quite different aspects. This results in the fact that there are many different equivalent axiomatizations, and it is sometimes very useful to "jump" from one point of view to another. Statements that are difficult to prove in one language may be easy in another. For this reason we present here several different axiomatizations. We also give a (partial) dictionary of how to translate among them. For a complete version of the basic equivalence proofs — which are highly non-trivial — see [3, Chapters 3 and 5].

We will give axiomatizations of oriented matroids for the following four types of representations:

- Collections of covectors,
- Collections of cocircuits,
- Signed bases,
- Arrangements of pseudospheres.

In the last part of this section these concepts are illustrated by an example.

GLOSSARY

Sign vector: Vector C in $\{-,0,+\}^E$, where E is a finite index set, usually $\{1,\dots,n\}$. For $e \in E$, the e-component of C is denoted by C_e.

Positive, negative, and zero part of C:

$$\begin{aligned} C^+ &:= \{e \in E \,:\, C_e = +\}, \\ C^- &:= \{e \in E \,:\, C_e = -\}, \\ C^0 &:= \{e \in E \,:\, C_e = 0\}. \end{aligned}$$

Support of C: $\underline{C} := \{e \in E \,:\, C_e \neq 0\}$.

Zero vector: $\mathbf{0} := (0,\dots,0) \in \{-,0,+\}^E$.

Negative of a sign vector: $-C$, defined by $(-C)^+ := C^-$, $(-C)^- := C^+$ and $(-C)^0 = C^0$.

Composition of C ***and*** D: $(C \circ D)_e := \begin{cases} C_e & \text{if } C_e \neq 0, \\ D_e & \text{otherwise.} \end{cases}$

Separation set of C ***and*** D: $S(C, D) := \{e \in E \,:\, C_e = -D_e \neq 0\}$.

We partially order the set of sign vectors by "$0 < +$" and "$0 < -$." The partial order on sign vectors, denoted by $C \leq D$, is understood componentwise; equivalently, we have

$$C \leq D \iff \Big[C^+ \subseteq D^+ \text{ and } C^- \subseteq D^- \Big].$$

For instance, if $C := (+, +, -, 0, -, +, 0, 0)$ and $D := (0, 0, -, +, +, -, 0, -)$, then we have:

$$C^+ = \{1, 2, 6\}, \quad C^- = \{3, 5\}, \quad C^0 = \{4, 7, 8\}, \quad \underline{C} = \{1, 2, 3, 5, 6\},$$

$$C \circ D = (+, +, -, +, -, +, 0, -), \quad C \circ D \geq C, \quad S(C, D) = \{5, 6\}.$$

Furthermore, for $x \in \mathbb{R}^n$, we denote by $\sigma(x) \in \{-, 0, +\}^E$ the image of x under the componentwise sign function σ that maps $\mathbb{R}^n$ to $\{-, 0, +\}^E$.

7.2.1 COVECTOR AXIOMS

DEFINITION (Covector Axioms): An ***oriented matroid*** given in terms of its covectors is a pair $\mathcal{M} := (E, \mathcal{L})$, where $\mathcal{L} \in \{-, 0, +\}^E$ satisfies

(CV0) $\mathbf{0} \in \mathcal{L}$,

(CV1) $C \in \mathcal{L} \implies -C \in \mathcal{L}$

(CV2) $C, D \in \mathcal{L} \implies C \circ D \in \mathcal{L}$

(CV3) $C, D \in \mathcal{L},\ e \in S(C, D) \implies$
there is a $Z \in \mathcal{L}$ with $Z_e = 0$ and with $Z_f = (C \circ D)_f$ for $f \in E \backslash S(C, D)$.

It is not difficult to check that these covector axioms are satisfied by the sign vector system $\mathcal{L}_X$ of the cells in a hyperplane arrangement $\mathcal{H}_X$, as defined in the last section. The first two axioms are satisfied trivially. For (CV2) assume that x_C and x_D are points in $\mathbb{R}^d$ with $\sigma(x_C^T \cdot X) = C \in \mathcal{L}_X$ and $\sigma(x_D^T \cdot X) = D \in \mathcal{L}_X$. Then (CV2) is implied by the fact that for sufficiently small $\varepsilon > 0$ we have $\sigma((x_C + \varepsilon x_D)^T \cdot X) = C \circ D$. The geometric content of (CV3) is that if $H_e := \{y \in \mathbb{R}^d \,:\, y^T x_e = 0\}$ is a hyperplane separating x_C and x_D then there exists a point x_Z on H_e with the property that x_Z is on the same side as x_C and x_D for all hyperplanes not separating x_C and x_D. We can find such a point by intersecting H_e with the line segment that connects x_C and x_D.

As we will see later the partially ordered set $(\mathcal{L}, \leq)$ describes the face lattice of a cell decomposition of the sphere S^{d-1} by pseudohyperspheres. Each sign vector corresponds to a face of the cell decomposition. We define the ***rank*** d of $\mathcal{M} = (E, \mathcal{L})$ to be the (unique) length of the maximal chains in $(\mathcal{L}, \leq)$ minus one. In the case of realizable arrangements $\mathcal{S}_X$ of hyperspheres, the lattice $(\mathcal{L}_X, \leq)$ equals the face lattice of $\Gamma(\mathcal{S}_X)$.

7.2.2 COCIRCUITS

The covectors of (inclusion) minimal support in $\mathcal{L}\backslash\mathbf{0}$ correspond to the 0-faces (=vertices) of the cell decomposition. We call the set $\mathcal{C}^*(\mathcal{M})$ of all such minimal covectors the ***cocircuits*** of $\mathcal{M}$. Oriented matroids can be described by their set of cocircuits, as shown by the following theorem.

THEOREM (Cocircuit characterization): *A collection $\mathcal{C}^* \in \{-,0,+\}^E$ is the set of cocircuits of an oriented matroid $\mathcal{M}$ if and only if it satisfies*

(CC0) $\mathbf{0} \notin \mathcal{C}^*$,

(CC1) $C \in \mathcal{C}^* \implies -C \in \mathcal{C}^*$

(CC2) *For all $C, D \in \mathcal{C}^*$ we have: $\underline{C} \subseteq \underline{D} \Longrightarrow C = D$ or $C = -D$.*

(CC3) *$C, D \in \mathcal{C}^*$, $C \neq -D$, and $e \in S(C,D) \implies$*
there is a $Z \in \mathcal{C}^*$ with $Z^+ \subseteq (C^+ \cup D^+)\backslash\{e\}$ and $Z^- \subseteq (C^- \cup D^-)\backslash\{e\}$.

THEOREM (Covector/cocircuit translation): *For every oriented matroid $\mathcal{M}$, one can uniquely determine the set $\mathcal{C}^*$ of cocircuits from the set $\mathcal{L}$ of covectors of $\mathcal{M}$, and conversely, as follows:*

(i) *$\mathcal{C}^*$ is the set of vectors with minimal support in $\mathcal{L}\backslash\mathbf{0}$:*
$\mathcal{C}^* = \{C \in \mathcal{L}\backslash\{\mathbf{0}\} \,:\, C' \leq C \Longrightarrow C' \in \{\mathbf{0}, C\}\}$

(ii) *$\mathcal{L}$ is the set of all sign vectors obtained by successive composition of a finite number of cocircuits from $\mathcal{C}^*$:*
$\mathcal{L} = \{C_1 \circ \ldots \circ C_k \,:\, k \geq 0,\ C_1, \ldots, C_k \in \mathcal{C}^*\}$.

7.2.3 CHIROTOPES

GLOSSARY

Alternating sign map: A map $\chi\colon E^d \longrightarrow \{-,0,+\}$ such that any transposition of two components changes the sign: $\chi(\tau_{ij}(\lambda)) = -\chi(\lambda)$.

Chirotope: An alternating sign map χ that encodes the basis orientations of an oriented matroid $\mathcal{M}$ of rank d.

We now present an axiom system for *chirotopes*, which characterizes oriented matroids in terms of basis orientations. Here an algebraic connection to determinant identities becomes obvious. Chirotopes are the main tool for translating problems in oriented matroid theory to an algebraic setting [7]. They also form a description of oriented matroids that is very practical for many algorithmic purposes (for instance in Computational Geometry, see Knuth [11]).

DEFINITION (Chirotope): Let $E := \{1, \ldots, n\}$ and $0 \leq d \leq n$. A ***chirotope of rank*** d is an alternating sign map $\chi: E^d \to \{-, 0, +\}$ that satisfies

(CHI1) The map $|\chi|: E^d \to \{0, 1\}$ which maps λ to $|\chi(\lambda)|$ is a matroid, and

(CHI2) For every $\lambda \in E^{d-2}$ and $a, b, c, d \in E \backslash \lambda$ the set

$$\Big\{\ \chi(\lambda, a, b) \cdot \chi(\lambda, c, d),\ -\chi(\lambda, a, c) \cdot \chi(\lambda, b, d),\ \chi(\lambda, a, d) \cdot \chi(\lambda, b, c)\ \Big\}$$

either contains $\{-1, +1\}$, or it equals $\{0\}$.

Where does the motivation of this axiomatization come from? If we again consider a configuration $X := (x_1, \ldots, x_n)$ of vectors in $\mathbb{R}^d$, we can observe the following identity among the $d \times d$ submatrices of X:

$$\begin{aligned} &\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_a, x_b) \cdot \det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_c, x_d) \\ -\ &\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_a, x_c) \cdot \det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_b, x_d) \\ +\ &\det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_a, x_d) \cdot \det(x_{\lambda_1}, \ldots, x_{\lambda_{d-2}}, x_b, x_c) \quad = \quad 0 \end{aligned}$$

for all $\lambda \in E^{d-2}$ and $a, b, c, d \in E \backslash \lambda$. Such a relation is called a *three-term Grassmann-Plücker identity.* If we compare this identity to our axiomatization, we see that (CHI2) implies that

$$\begin{array}{rrcl} \chi_X: & E^d & \to & \{-, 0, +\} \\ & (\lambda_1, \ldots, \lambda_d) & \mapsto & \mathrm{sign}(\det(x_{\lambda_1}, \ldots, x_{\lambda_d})). \end{array}$$

is not in contradiction to these identities. More precisely, if we consider χ_X as defined above for a vector configuration X, the above Grassmann-Plücker identities imply that (CHI2) is satisfied. (CHI1) is also satisfied since for the vectors of X the Steinitz exchange axiom holds. (In fact the exchange axiom is a consequence of higher order Grassmann-Plücker identities.)

Consequently, χ_X is a chirotope for every $X \in (\mathbb{R}^d)^n$. Thus chirotopes can be considered as a combinatorial model of the determinant values on vector configurations. The following is not easy to prove, but essential.

THEOREM (Chirotope/cocircuit translation): *For each chirotope χ of rank d on $E := \{1, \ldots, n\}$ the set*

$$\mathcal{C}^*(\chi) \ = \ \Big\{ \big(\chi(\lambda, 1), \chi(\lambda, 2), \ldots, \chi(\lambda, n)\big) \ : \ \lambda \in E^{d-1} \Big\}$$

forms the set of cocircuits of an oriented matroid. Conversely, for every oriented matroid $\mathcal{M}$ with cocircuits $\mathcal{C}^*$ there exists a unique pair of chirotopes $\{\chi, -\chi\}$ such that $\mathcal{C}^*(\chi) = \mathcal{C}^*(-\chi) = \mathcal{C}^*$.

The retranslation of cocircuits into signs of bases is straightforward but needs extra notation. It is omitted here.

7.2.4 ARRANGEMENTS OF PSEUDOSPHERES

GLOSSARY

The $(d-1)$-sphere: The standard unit sphere $S^{d-1} := \{x \in \mathbb{R}^d \ : \ ||x|| = 1\}$, or any homeomorphic image of it.

Pseudosphere: The image $s \subseteq S^{d-1}$ of the equator $\{x \in S^{d-1} \ : \ x_d = 0\}$ in the unit sphere under a self homeomorphism $\phi: S^{d-1} \to S^{d-1}$.
(This definition describes topologically *tame* embeddings of a $(d-2)$-sphere in S^{d-1}. Pseudospheres behave "nicely" in the sense that they divide S^{d-1} into two sides homeomorphic to open $(d-1)$-balls.)

Oriented pseudosphere: A pseudosphere together with a choice of a positive side s^+ and a negative side s^-.

Arrangement of pseudospheres: A set of n pseudospheres in S^{d-1} with the extra condition that any subset of $d+2$ or fewer pseudospheres is ***realizable***: it defines a cell decomposition of S^{d-1} that is isomorphic to a decomposition by an arrangement of $d+2$ linear hyperplanes.

Essential arrangement: Arrangement such that the intersection of all the pseudospheres is empty.

Rank: The codimension in S^{d-1} of the intersection of all the pseudospheres. For an essential arrangement in S^{d-1}, the rank is d.

Topological representation of $\mathcal{M} = (E, \mathcal{L})$: An essential arrangement of oriented pseudospheres such that $\mathcal{L}$ is the collection of sign vectors associated with the cells of the arrangement.

One of the most important interpretations of oriented matroids is given by the Topological Representation Theorem of Folkman & Lawrence [**9**] [**3**, Chapters 4 and 5]. It states that oriented matroids are in bijection to (combinatorial equivalence classes of) *arrangements of oriented pseudospheres*. Arrangements of pseudospheres are a topological generalization of hyperplane arrangements, in the same way in which arrangements of pseudolines generalize line arrangements. Thus every rank d oriented matroid describes a certain cell decomposition of the $(d-1)$-sphere. Arrangements of pseudospheres are collections of pseudospheres that have intersection properties just like those satisfied by arrangements of proper subspheres.

DEFINITION (Arrangement of Pseudospheres): A finite collection $\mathcal{P} = (s_1, s_2, \ldots, s_n)$ of pseudospheres in S^{d-1} is an *arrangement of pseudospheres* if the following conditions hold (we set $E := \{1, \ldots, n\}$):

(PS1) For all $A \subseteq E$ the set $S_A = \bigcap_{e \in A} s_e$ is a topological sphere.

(PS2) If $S_A \not\subseteq s_e$, for $A \subseteq E, e \in E$, then $S_A \cap s_e$ is a pseudosphere in S_A with sides $S_A \cap s_e^+$ and $S_A \cap s_e^-$.

Notice that this definition permits two pseudospheres of the arrangement to be identical. An entirely different, but equivalent, definition is given in the Glossary.

We see that every essential arrangement of pseudospheres $\mathcal{P}$ partitions the $(d-1)$-sphere into a regular cell complex $\Gamma(\mathcal{P})$. Each cell of $\Gamma(\mathcal{P})$ is uniquely determined by a sign vector in $\{-, 0, +\}^E$ encoding the relative position with respect to each pseudosphere s_i. Conversely, $\Gamma(\mathcal{P})$ characterizes $\mathcal{P}$ up to homeomorphism. $\mathcal{P}$ is *realizable* if there exists an arrangement of proper spheres $\mathcal{S}_X$ with $\Gamma(\mathcal{P}) \cong \Gamma(\mathcal{S}_X)$.

The translation of arrangements of pseudospheres to oriented matroids is given by the Topological Representation Theorem of Folkman and Lawrence [9]. (For the definition of "loop," see Section 7.3.1.)

THEOREM (Topological Representation Theorem [9] **— Pseudosphere-covector translation):** *If $\mathcal{P}$ is an essential arrangement of pseudospheres on S^{d-1} then $\Gamma(\mathcal{P}) \cup \mathbf{0}$ forms the set of covectors of an oriented matroid of rank d. Conversely, for every oriented matroid $(E, \mathcal{L})$ of rank d (without loops) there exists an essential arrangement of pseudospheres $\mathcal{P}$ on S^{d-1} with $\Gamma(\mathcal{P}) = \mathcal{L} \backslash \mathbf{0}$.*

7.2.5 DUALITY

GLOSSARY

Orthogonality: Two sign vectors $C, D \in \{-, 0, +\}^E$ are *orthogonal* if the set

$$\{C_e \cdot D_e \ : \ e \in E\}$$

either equals $\{0\}$ or contains both $+$ and $-$. We then write $C \perp D$.

Vector of $\mathcal{M}$: Sign vector that is orthogonal to all covectors of $\mathcal{M}$; covector of the dual oriented matroid $\mathcal{M}^*$.

Circuit of $\mathcal{M}$: Vector of minimal nonempty support; cocircuit of the dual oriented matroid $\mathcal{M}^*$.

There is a natural duality structure relating oriented matroids of rank d on n elements to oriented matroids of rank $n-d$ on n elements. It is an amazing fact that the existence of such a duality relation can be used to give another axiomatization of oriented matroids (see [3, Sect. 3.4]). Here we restrict ourselves to the definition of the dual of an oriented matroid $\mathcal{M}$.

THEOREM (Duality): *For every oriented matroid $\mathcal{M} = (E, \mathcal{L})$ of rank d there is a unique oriented matroid $\mathcal{M}^* = (E, \mathcal{L}^*)$ of rank $|E| - d$ given by*

$$\mathcal{L}^* \quad = \quad \Big\{ D \in \{-, 0, +\}^E \ : \ C \perp D \text{ for every } C \in \mathcal{L} \Big\}$$

*$\mathcal{M}^*$ is called the **dual of** $\mathcal{M}$. In particular, $(\mathcal{M}^*)^* = \mathcal{M}$.*

In particular, the cocircuits of the dual oriented matroid $\mathcal{M}^*$, which we call the *circuits* of $\mathcal{M}$, also determine $\mathcal{M}$. Hence the collection $\mathcal{C}(\mathcal{M})$ of all circuits of an oriented matroid $\mathcal{M}$, given by

$$\mathcal{C}(\mathcal{M}) \quad := \quad \mathcal{C}^*(\mathcal{M}^*).$$

is characterized by the *the same* cocircuit axioms. Analogously, the ***vectors*** of $\mathcal{M}$ are obtained as the covectors of $\mathcal{M}^*$; they are characterized by the covector axioms.

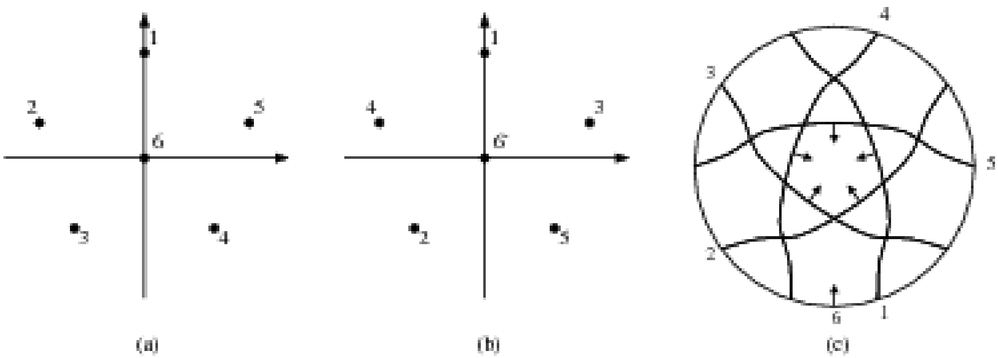
An oriented matroid $\mathcal{M}$ is realizable if and only if its dual $\mathcal{M}^*$ is realizable. The reason for this is that a matrix $(I_d|A)$ represents $\mathcal{M}$ if and only if $(-A^T|I_{n-d})$ represents $\mathcal{M}^*$. (Here I_d denotes a $d \times d$ identity matrix, $A \in \mathbb{R}^{d\times(n-d)}$, and $A^T \in \mathbb{R}^{(n-d)\times d}$ denotes the transpose of A.)

Thus for a realizable oriented matroid $\mathcal{M}_X$ the vectors represent the linear dependencies among the columns of X, while the circuits represent minimal linear dependencies. Similarly, in the pseudoarrangements picture, circuits correspond to minimal systems of closed hemispheres that cover the whole sphere, while vectors correspond to consistent unions of such covers that never require the use of both hemispheres determined by a pseudosphere. This provides a direct geometric interpretation of circuits and vectors.

7.2.6 AN EXAMPLE

We close this section with an example that demonstrates the different representations of an oriented matroid. Consider the planar point configuration X given in Figure 7.3(a).

Figure 7.3
An example of an oriented matroid on 6 elements.



Homogeneous coordinates for X are given by

$$X := \begin{pmatrix} 0 & 3 & 1 \\ -3 & 1 & 1 \\ -2 & -2 & 1 \\ 2 & -2 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The chirotope χ_X of $\mathcal{M}$ is given by the orientations:

$$\begin{array}{lllll}
\chi(1,2,3) = + & \chi(1,2,4) = + & \chi(1,2,5) = + & \chi(1,2,6) = + & \chi(1,3,4) = + \\
\chi(1,3,5) = + & \chi(1,3,6) = + & \chi(1,4,5) = + & \chi(1,4,6) = - & \chi(1,5,6) = - \\
\chi(2,3,4) = + & \chi(2,3,5) = + & \chi(2,3,6) = + & \chi(2,4,5) = + & \chi(2,4,6) = + \\
\chi(2,5,6) = - & \chi(3,4,5) = + & \chi(3,4,6) = + & \chi(3,5,6) = + & \chi(4,5,6) = +
\end{array}$$

Half of the cocircuits of $\mathcal{M}$ are given in the table below (the other half is obtained by negating the data):

$$\begin{array}{lll}
(0,0,+,+,+,+) & (0,-,0,+,+,+) & (0,-,-,0,+,-) \\
(0,-,-,-,0,-) & (0,-,-,+,+,0) & (+,0,0,+,+,+) \\
(+,0,-,0,+,+) & (+,0,-,-,0,-) & (+,0,-,-,+,0) \\
(+,+,0,0,+,+) & (+,+,0,-,0,+) & (+,+,0,-,-,0) \\
(+,+,+,0,0,+) & (-,+,+,0,-,0) & (-,-,+,+,0,0)
\end{array}$$

Observe that the cocircuits correspond to the point partitions produced by hyperplanes spanned by points. Half of the circuits of $\mathcal{M}$ are given in the next table. The circuits correspond to sign patterns induced by minimal linear dependencies on the rows of the matrix X. It is easy to check that every pair consisting of a circuit and a cocircuits fulfills the orthogonality condition.

$$\begin{array}{lll}
(+,-,+,-,0,0) & (+,-,+,0,-,0) & (+,-,+,0,0,-) \\
(+,-,0,+,-,0) & (+,+,0,+,0,-) & (+,-,0,0,-,+) \\
(+,0,-,+,-,0) & (+,0,+,+,0,-) & (+,0,+,0,+,-) \\
(+,0,0,+,-,-) & (0,+,-,+,-,0) & (0,+,-,+,0,-) \\
(0,+,+,0,+,-) & (0,+,0,+,+,-) & (0,0,+,-,+,-)
\end{array}$$

An affine picture of a realization of the dual oriented matroid is given in Figure 7.3.(b). The minus-sign at point **6** indicates that a reorientation at point **6** has taken place. It is easy to check that the circuits and the cocircuits interchange their role when dualizing the oriented matroid.

Figure 7.3(c) shows the corresponding arrangement of pseudolines. The circle bounding the configuration represents the projective line at infinity representing line **6**.

7.3 IMPORTANT CONCEPTS

In this section we briefly introduce some very basic concepts in the theory of oriented matroids. The list of topics treated here is tailored towards some areas of oriented matroid theory that are particularly relevant for applications. Thus many other topics of great importance are left out. In particular, see [3, Sect. 3.3] for minors of oriented matroids, and [3, Chap. 7] for basic constructions.

7.3.1 SOME VERY BASIC CONCEPTS

In the following glossary, we list some fundamental concepts of oriented matroid theory. Each of them can be expressed in terms of any one of the representations of oriented matroids that we have introduced (covectors, cocircuits, chirotopes, pseudoarrangements), but for each of these concepts some representations are much more convenient than others. Also, each of these concepts has some interesting properties with respect to the duality operator — which may be more or less obvious, depending on the representation that one uses.

GLOSSARY

Direct sum: An oriented matroid $\mathcal{M} = (E, \mathcal{L})$ has a ***direct sum decomposition***, denoted by $\mathcal{M} = \mathcal{M}(E_1) \oplus \mathcal{M}(E_2)$, if E has a partition into nonempty subsets E_1 and E_2 such that $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ for two oriented matroids $\mathcal{M}_1 = (E_1, \mathcal{L}_1)$ and $\mathcal{M}_2 = (E_2, \mathcal{L}_2)$. If $\mathcal{M}$ has no direct sum decomposition, then it is ***irreducible***.

Loops and coloops: A loop of $\mathcal{M} = (E, \mathcal{L})$ is an element $e \in E$ that satisfies $C_e = 0$ for all $C \in \mathcal{L}$. A coloop satisfies $\mathcal{L} \cong \mathcal{L}' \times \{-, 0, +\}$, where $\mathcal{L}'$ is obtained by deleting the e-components from the vectors in $\mathcal{L}$.
If $\mathcal{M}$ has a direct sum decomposition with $E_2 = \{e\}$, then e is either a loop or a coloop.

Acyclic: An oriented matroid $\mathcal{M} = (E, \mathcal{L})$ for which $(+, \dots, +) \in \mathcal{L}$ is a covector; equivalently, the union of the supports of all non-negative cocircuits is E.

Totally cyclic: An oriented matroid without non-negative cocircuits; equivalently, $\mathcal{L} \cap \{0, +\}^E = \{\mathbf{0}\}$.

Uniform: An oriented matroid $\mathcal{M}$ of rank d on E is ***uniform*** if all of its cocircuits have size $|E| - d + 1$.
Equivalently, $\mathcal{M}$ is uniform if it has a chirotope with values in $\{+, -\}$.

$\mathcal{M}$ is realizable: There is a vector configuration X with $\mathcal{M}_X = \mathcal{M}$.

Realization of $\mathcal{M}$: A vector configuration X with $\mathcal{M}_X = \mathcal{M}$.

THEOREM (Duality II): *Let $\mathcal{M}$ be an oriented matroid on the ground set E, and $\mathcal{M}^*$ its dual.*

- *$\mathcal{M}$ is acyclic if and only if $\mathcal{M}^*$ is totally cyclic. (However, "most" oriented matroids are neither acyclic nor totally cyclic!)*
- *$e \in E$ is a loop of $\mathcal{M}$ if and only if it is a coloop of $\mathcal{M}^*$.*
- *$\mathcal{M}$ is uniform if and only if $\mathcal{M}^*$ is uniform.*
- *$\mathcal{M}$ is a direct sum $\mathcal{M}(E) = \mathcal{M}(E_1) \oplus \mathcal{M}(E_2)$ if and only if $\mathcal{M}^*$ is a direct sum $\mathcal{M}^*(E) = \mathcal{M}^*(E_1) \oplus \mathcal{M}^*(E_2)$.*

Duality of oriented matroids captures, among other things, the concepts of linear programming duality [1] [3, Chap. 10] and the concept of Gale diagrams for polytopes [19, Lect. 6]. For the latter, we note here that the vertex set of a d-dimensional convex polytope P with $d+k$ vertices yields a configuration of $d+k$ vectors in $\mathbb{R}^{d+1}$, and thus an oriented matroid of rank $d+1$ on $d+k$ points. Its dual is a realizable oriented matroid of rank $k-1$, the *Gale diagram* of P. It can be modelled by an affine point configuration of dimension $k-2$, called an *affine Gale diagram* of P. Hence, for "small" k, we can represent a (possibly high-dimensional) polytope with "few vertices" by a low-dimensional point configuration. In particular, this is beneficial in the case $k = 4$, where polytopes with "universal" behaviour can be analyzed in terms of their 2-dimensional affine Gale diagrams. For further details, see Chapter 14 of this Handbook.

7.3.2 REALIZABILITY AND REALIZATION SPACES

GLOSSARY

Realization space: Let $\chi: E^d \to \{-,0,+\}$ be a chirotope with $\chi(1,\ldots,d) = +$. The *realization space* $\mathcal{R}(\chi)$ is the set of all matrices $X \in \mathbb{R}^{d\cdot n}$ with $\chi_X = \chi$ and $x_i = e_i$ for $i = 1,\ldots,d$, where e_i is the i-th unit vector. If $\mathcal{M}$ is the corresponding oriented matroid, we write $\mathcal{R}(\mathcal{M}) = \mathcal{R}(\chi)$.

Rational realization: A realization $X \in \mathbb{Q}^{d\cdot n}$; that is, a point in $\mathcal{R}(\chi) \cap \mathbb{Q}^{d\cdot n}$.

Primary semialgebraic set: The (real) solution set of an arbitrary finite system of polynomial equations and strict inequalities with integer coefficients.

"Existential Theory of the Reals": The problem of solving arbitrary systems of polynomial equations and inequalities with integer coefficients.

Stable equivalence: A strong type of arithmetic and homotopy equivalence. Two semialgebraic sets are stably equivalent if they can be connected by a sequence of rational coordinate changes, together with certain projections with contractible fibers. (See [17] for details).
In particular, two stably equivalent semialgebraic sets have the same number of components, they are homotopy equivalent, and either both or neither of them have rational points.

It is one of the most exciting problems in oriented matroid theory to design algorithms that find a realization of a given oriented matroid if it exists. However, for oriented matroids with a large numbers of points one cannot be too optimistic, since the realizability problem for oriented matroids is NP-hard. This is one of the consequences of Mnëv's Universality Theorem below. An upper bound for the worst case complexity of the realizability problem is given by the following theorem. It follows from general complexity bounds for algorithmic problems about semialgebraic sets by Basu, Pollack & Roy [2] (see also Chapter 30 of this Handbook).

THEOREM (Complexity of the best general algorithm known): *The realizability of a rank d oriented matroid on n points can be decided by solving a system of $S = \binom{n}{d}$ real polynomial equations and strict inequalities of degree at most $D = d-1$ in $K = (n-d-1)(d-1)$ variables. Thus, with the algorithms of [2] the number of bit operations needed to decide realizability is (in the Turing machine model of complexity) bounded by $(S/K)^K \cdot S \cdot D^{O(K)}$.*

THE UNIVERSALITY THEOREM

A basic observation is that all oriented matroids of rank 2 are realizable. In particular, up to change of orientations and permuting the elements in E there is only one uniform oriented matroid of rank 2. The realization space of an oriented matroid of rank 2 is always stably equivalent to some $\mathbb{R}^K$; in particular, if $\mathcal{M}$ is uniform of rank 2 on n elements, then $\mathcal{R}(\mathcal{M})$ is isomorphic to an open subset of $\mathbb{R}^{2n-4}$.

In contrast to the rank 2 case, Mnëv's Universality Theorem states that for oriented matroids of rank 3, the realization space can be "arbitrarily complicated." Here is the first glimpse of this:

- The realization spaces of all realizable uniform oriented matroids of rank 3 and at most 9 elements are contractible (Richter [16]).
- There is a realizable rank 3 oriented matroid on 9 elements that has no realization with rational coordinates (Perles).
- There is a realizable rank 3 oriented matroid on 14 elements with disconnected realization space (Suvorov).

The Universality Theorem is a fundamental statement with various implications for the configuration spaces of various types of combinatorial objects.

THEOREM (Mnëv's Universality Theorem [14]): *For every primary semi-algebraic set V defined over $\mathbb{Z}$ there is a chirotope χ of rank 3 such that V and $\mathcal{R}(\chi)$ are stably equivalent.*

Although some of the facts in the following list were proved earlier than Mnëv's Universality Theorem, they all can be considered as consequences of the construction techniques used by Mnëv.

CONSEQUENCES of the Universality Theorem

- The full field of algebraic numbers is needed to realize all oriented matroids of rank 3.
- The realizability problem for oriented matroids is NP-hard (Mnëv, Shor).
- The realizability problem for oriented matroids is (polynomial time) equivalent to the "Existential Theory of the Reals" (Mnëv).
- For every finite simplicial complex Δ, there is an oriented matroid whose realization space is homotopy equivalent to Δ.
- Realizability of rank 3 oriented matroids cannot be characterized by excluding a finite set of "forbidden minors" (Bokowski & Sturmfels).
- In order to realize all combinatorial types of integral rank 3 oriented matroids on n elements, even uniform ones, in the integer grid $\{1, 2, \ldots, f(n)\}^3$, the "coordinate size" function $f(n)$ has to grow doubly exponentially in n (Goodman, Pollack & Sturmfels).
- The "isotopy problem" for oriented matroids (Can one given realization of $\mathcal{M}$ be continuously deformed, through realizations, to another given one?) has a negative solution in general, even for uniform oriented matroids of rank 3.

7.3.3 TRIANGLES AND SIMPLICIAL CELLS

There is a long tradition of studying *triangles* in arrangements of pseudolines. Already Levi in his 1926 paper [13] considered them to be important structures.

There are good reasons for this. On the one hand, they form the simplest possible cells of full dimension, and are therefore of basic interest. On the other hand, if the arrangement is simple, triangles locate the regions where a "smallest" local change of the combinatorial type of the arrangement is possible. Such a change can be performed by taking one side of the triangle and "pushing" it over the vertex formed by the other two sides. It was observed by Ringel [18] that any two simple arrangements of pseudolines can be deformed into one another by performing a sequence of such "triangle flips."

Moreover, the realizability of a pseudoline arrangement may depend on the situation at the triangles. For instance, if any one of the triangles in the non-realizable example of Figure 7.2 other than the central one is flipped, the whole configuration becomes realizable.

MAIN RESULTS on triangles in arrangements of pseudolines

Let $\mathcal{P}$ be any arrangement of n pseudolines.

- For any pseudoline ℓ in $\mathcal{P}$ there are at least 3 triangles adjacent to ℓ. Either the $n-1$ pseudolines different from ℓ intersect in one point (i.e., $\mathcal{P}$ is a ***near-pencil***), or there are at least $n-3$ triangles that are not adjacent to ℓ. Thus $\mathcal{P}$ contains at least n triangles (Levi).
- $\mathcal{P}$ is ***simplicial*** if all its regions are bounded by exactly 3 (pseudo)lines. Except for the near-pencils, there are two infinite classes of simplicial line arrangements and 91 additional "sporadic" simplicial line arrangements (and many more simplicial pseudoarrangements) known (Grünbaum).
- If $\mathcal{P}$ is simple, then it contains at most $\frac{n(n-1)}{3}$ triangles. For infinitely many n there exists a simple arrangement with $\frac{n(n-1)}{3}$ triangles (Roudneff, Harborth).
- Any two simple arrangements $\mathcal{P}_1$ and $\mathcal{P}_2$ can be deformed into one another by a sequence of simplicial flips (Ringel [18]).

Every arrangement of pseudospheres in S^{d-1} has a centrally symmetric representation. Thus we can always derive an arrangement of projective pseudohyperplanes (pseudo $(d-2)$-planes in $\mathbb{R}P^{d-1}$) by identifying antipodal points. The proper analogue for the triangles in rank 3 are the $(d-1)$-simplices in projective arrangements of pseudohyperplanes in rank d, i.e., the regions bounded by the minimal number of d pseudohyperplanes. We call an arrangement *simple* if no more than $d-1$ planes meet in a point.

It was conjectured by Las Vergnas in 1980 [12] that (as in the rank 3 case) any two simple arrangements can be transformed into each other by a sequence of flips of simplicial regions. In particular this requires that every simple arrangement contains *at least one* simplicial region (which was also conjectured by Las Vergnas). If we consider the case of realizable arrangements only, it is not difficult to prove that any two members in this subclass can be connected by a sequence of flips of simplicial regions and that each realizable arrangement contains at least one simplicial cell. In fact, Shannon proved that every arrangement (even the non-simple) of n projective hyperplanes in rank d contains at least n simplicial regions.

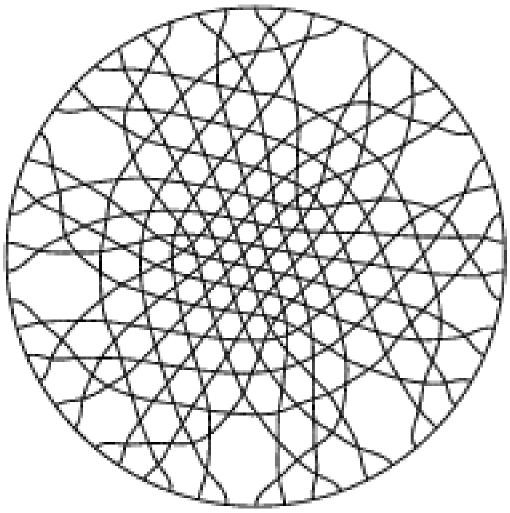


Figure 7.4
A simple arrangement of 28 *pseudolines with a maximal number of* 252 *triangles.*

More precisely, for every hyperplane h there are at least d simplices adjacent to h and at least $n-d$ simplices not adjacent to h. The contrast between the Las Vergnas conjecture and the results known for the non-realizable case is dramatic:

MAIN RESULTS on simplicial cells in pseudoarrangements

- There is an arrangement of 8 pseudoplanes in rank 4 having only 7 simplicial regions (Roudneff & Sturmfels, Altshuler & Bokowski).
- Every rank 4 arrangement with $n < 13$ pseudoplanes has at least one simplicial region (Bokowski).
- For every $k > 2$ there is a rank 4 arrangement of $4k$ pseudoplanes having only $3k+1$ simplicial regions (Richter-Gebert).
- There is a rank 4 arrangement consisting of 20 pseudoplanes for which one plane is not adjacent to any simplicial region (Richter-Gebert).

OPEN PROBLEMS

The topic of simplicial cells is interesting and rich in structure even in rank 3. The case of higher dimensions is full of unsolved problems and challenging conjectures. These problems are relevant for various problems of great geometric and topological interest, such as the structure of spaces of triangulations. Three key problems are:

- Classify simplical arrangements. Is it true, at least, that there are only finitely many types of simplicial arrangements of straight lines outside the three known infinite families?

- Does every arrangement of pseudohyperplanes contain at least one simplicial region?
- Is it true that any two simple arrangements can be transformed into one another by a sequence of flips of simplicial regions?

7.3.4 MATROID POLYTOPES

The convexity properties of a point configuration X are modelled superbly by the oriented matroid $\mathcal{M}_X$. The combinatorial versions of many theorems concerning convexity also hold on the level of general (including non-realizable) oriented matroids. For instance, there are purely combinatorial versions of Carathedory's, Radon's, and Helly's Theorems [**3**, Sect. 9.2].

In particular, oriented matroid theory provides us with an entirely combinatorial model of convex polytopes, known as "matroid polytopes." The following definition provides this context in terms of face lattices.

DEFINITION (Matroid polytope): The face lattice of an acyclic oriented matroid $\mathcal{M} = (E, \mathcal{L})$ is the set

$$\mathrm{FL}(\mathcal{M}) := \{C^0 \ : \ C \in \mathcal{L} \cap \{0,+\}^E\},$$

partially ordered by inclusion. The elements of $\mathrm{FL}(\mathcal{M})$ are the ***faces*** of M. $\mathcal{M}$ is a ***matroid polytope*** if $\{e\}$ is a face for every $e \in E$.

Every polytope gives rise to a matroid polytope: if $P \subseteq \mathbb{R}^d$ is a d-polytope with n vertices, then the canonical embedding $x \mapsto \binom{x}{1}$ creates a vector configuration X_P of rank $d{+}1$ from the vertex set of P. The oriented matroid of X_P is a matroid polytope $\mathcal{M}_P$, whose face lattice $\mathrm{FL}(\mathcal{M})$ is canonically isomorphic to the face lattice of P.

Matroid polytopes provide a very precise model of (the combinatorial structure of) convex polytopes. In particular, the Topological Representation Theorem implies that *every* matroid polytope of rank d is the face lattice of a regular piecewise linear (PL) cell decomposition of a $(d{-}2)$-sphere. Thus matroid polytopes form an excellent combinatorial model for convex polytopes: in fact, much better than the model of PL spheres (which does not have an entirely combinatorial definition).

However, the construction of a polar fails in general for matroid polytopes. The cellular spheres that represent matroid polytopes have dual cell decompositions (because they are piecewise linear), but this dual cell decomposition is not in general a matroid polytope, even in rank 4. (Billera & Munson; Bokowski & Schuchert [**6**]). In other words, the order dual of the face lattice of a matroid polytope (as an abstract lattice) is *not in general* the face lattice of a matroid polytope. (Matroid polytopes form an important tool for polytope theory, not only because of the parts of polytope theory that work for them, but also because of those that fail.)

For every matroid polytope one has the dual oriented matroid (which is totally cyclic, hence not a matroid polytope). In particular, the set-up for Gale diagrams generalizes to the framework of matroid polytopes; this makes it possible to also include non-polytopal spheres in a discussion of the realizability properties of polytopes. This amounts to perhaps the most powerful single tool ever developed for polytope theory. It leads to, among other things, the classification of d-dimensional polytopes with at most $d+3$ vertices, the proof that all matroid polytopes of rank $d+1$ with at most $d+3$ vertices are realizable, the construction of non-rational polytopes, as well as of non-polytopal spheres with $d+4$ vertices, etc.

ALGORITHMIC APPROACH to the classification of polytopes

A powerful approach, via matroid polytopes, to the problem of classifying all convex polytopes with given parameters is largely due to Bokowski and Sturmfels [7]. Here we restrict our attention to the simplicial case — there are additional technical problems to deal with in the non-simplicial case, and very little work has been done there, as yet. However, the program has been successfully completed for the classification of all simplicial 3-spheres with 9 vertices (Altshuler, Bokowski & Steinberg) and of all neighborly 5-spheres with 10 vertices (Bokowski & Shemer) into polytopes and non-polytopes. At the core of the matroidal approach lies the following hierarchy:

$$\begin{pmatrix}\text{simplicial}\\ \text{spheres}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{uniform}\\ \text{matroid polytopes}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{convex}\\ \text{polytopes}\end{pmatrix}.$$

The plan of attack is the following. First, one enumerates all isomorphism types of simplicial spheres with given parameters. Then, for each sphere, one computes *all* (uniform) matroid polytopes that have the given sphere as their face lattices. Finally, for each matroid polytope, one tries to decide realizability.

At both of the steps of this hierarchy there are considerable subtleties involved that lead to important insights. For a given simplicial sphere, there may be

- *no* matroid polytope that supports it. In this case the sphere is called ***non-matroidal***. The Barnette sphere [3, Prop. 9.5.3] is an example.
- *exactly one* matroid polytope. In this (important) case the sphere is called ***rigid***. That is, a matroid polytope $\mathcal{M}$ is rigid if $\mathrm{FL}(\mathcal{M}') = \mathrm{FL}(\mathcal{M})$ already implies $\mathcal{M}' = \mathcal{M}$. For rigid matroid polytopes the face lattice uniquely defines the oriented matroid, and thus every statement about the matroid polytope yields a statement about the sphere. In particular, the matroid polytope and the sphere have the same realization space.

  Rigid matroid polytopes are a priori rare; however, the *Lawrence construction* [3, Sec. 9.3] [19, Sect. 6.6] associates with every oriented matroid $\mathcal{M}$ on n elements in rank d a rigid matroid polytope $\Lambda(\mathcal{M})$ with $2n$ vertices of rank $n+d$. The realizations of $\Lambda(\mathcal{M})$ can be retranslated into realizations of $\mathcal{M}$.
- or *many* matroid polytopes.

The situation is similarly complex for the second step, from matroid polytopes to convex polytopes. In fact, for each matroid polytope there may be

- *no* convex polytope — this is the case for a non-realizable matroid polytope. These exist already with relatively few vertices; namely in rank 5 with 9 vertices [6], and in rank 4 with 10 vertices [3, Prop. 9.4.5].
- essentially *only one* — this is the rare case where the matroid polytope is "projectively unique."

- or *many* convex polytopes — the space of all polytopes for a given matroid polytope is the realization space of the oriented matroid, and this may be arbitrarily complicated. In fact, a combination of Mnëv's Universality Theorem, the Lawrence construction, and a scattering technique [7, Thm 6.2] (in order to obtain the simplicial case) yields the following amazing Universality Theorem.

THEOREM (Mnëv's Universality Theorem for Polytopes [14]): *For every [open] primary semialgebraic set V defined over $\mathbb{Z}$ there is an integer d and a [simplicial] d-dimensional polytope P on $d+4$ vertices such that V and the realization space of P are stably equivalent.*

RELATED CHAPTERS

Chapter 6: Pseudoline arrangements

Chapter 14: Basic properties of convex polytopes

Chapter 22: Arrangements

Chapter 30: Computational real algebraic geometry

Chapter 40: Mathematical programming

Chapter 49: Geometric applications of the Cayley-Grassmann algebra

FURTHER READING

The basic theory of oriented matroids was introduced in two fundamental papers, Bland & Las Vergnas [4] and Folkman & Lawrence [9]. We refer to the monograph by Björner, Las Vergnas, Sturmfels, White & Ziegler [3] for a broad introduction, and for an extensive development of the theory of oriented matroids with a reasonably complete bibliography. Other introductions and basic sources of information include Bachem & Kern [1], Bokowski [5], Bokowski & Sturmfels [7], and Ziegler [19, Lect. 6 and 7].

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