Shelling Polyhedral 3-Balls and 4-Polytopes

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Abstract

There is a long history of constructions of non-shellable triangulations of 3-dimensional (topological) balls. This paper gives a survey of these constructions, including Furch's 1924 construction using knotted curves, which also appears in Bing's 1962 survey of combinatorial approaches to the Poincaré conjecture, Newman's 1926 explicit example, and M. E. Rudin's 1958 non-shellable triangulation of a tetrahedron with only 14 vertices (all on the boundary) and 41 facets. Here an (extremely simple) new example is presented: a non-shellable simplicial 3-dimensional ball with only 10 vertices and 21 facets.

It is further shown that shellings of simplicial 3-balls and 4-polytopes can "get stuck": simplicial 4-polytopes are not in general "extendably shellable." Our constructions imply that a Delaunay triangulation algorithm of Beichl & Sullivan, which proceeds along a shelling of a Delaunay triangulation, can get stuck in the 3D version: for example, this may happen if the shelling follows a knotted curve.

1 Introduction

Shellability is a concept that has its roots in the theory of convex polytopes, going back to Schläfli's work [30] written in 1852. The power of shellability became apparent with Bruggesser & Mani's 1971 proof [14] that all convex polytopes are shellable: this provided a very simple proof for the d-dimensional Euler-Poincaré formula, and it lies at the heart of McMullen's 1970 proof [26] of the Upper Bound Theorem. The method has found similarly striking applications in Computational Geometry (see Seidel [31], and the work of Beichl & Sullivan [2, 3] discussed below), and also in purely combinatorial problems (see Björner [5, 6], and Björner & Wachs [9, 10] for recent work on lexicographic shellability and applications).

However, in the realm of low-dimensional topology (without a convexity assumption) shellability is usually hard to establish, and remaims elusive: see for example Frankl's [19], Zeeman's [34], and Bing's [4] work on the 3-dimensional Poincaré conjecture.

This paper focuses on the *geometry* of non-shellable complexes, and thus tries to relate the convexity aspects with the topological difficulties. Besides a brief survey of non-shellable balls and spheres — meant to complement Danaraj & Klee's [17] account from 1978, and the treatment in [35, Lect. 8] — we present the currently "smallest" example of a non-shellable ball.

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We then discuss *extendable shellability*. This concept is essential for the algorithmic use of shellings, as pointed out by Danaraj & Klee [17]. In [35, Sect. 8.3] we have shown that in contrast to a conjecture by Tverberg [17, p. 37], not all polytopes are extendably shellable: shellings can "get stuck." Here we show, based on Bing's knotted holes, that for $d \geq 4$ "most" convex 4-polytopes are not extendably shellable. In particular, neither simplicial polytopes, nor simple polytopes, are extendably shellable in general. The construction also implies that, without extra provisions, algorithms such as the Beichl-Sullivan procedure to compute 3-dimensional Delaunay triangulations may fail.

2 Basic Definitions

Definition 2.1 (See [35, Sect. 5.1])
A **polytopal complex** *is a finite collection* Γ *of polytopes, including* $\emptyset \in \Gamma$*, in some real vector space* $\mathbb{R}^d$ *such that*

(i) *every face of a polytope* $P \in \Gamma$ *is also contained in* Γ*,*

(ii) *the intersection* $P \cap Q$ *of two polytopes* $P, Q \in \Gamma$ *is a (possibly empty) face of each of them.*

In this definition, we consider $\emptyset$ to be a polytope "of dimension -1." In part (ii), we automatically get $P \cap Q \in \Gamma$, by (i): this intersection may be empty.

All complexes Γ that we consider in the following are *pure*, that is, they satisfy the condition that all the maximal faces with respect to inclusion, called the *facets* of Γ, have the same dimension, called the *dimension of* Γ. (See Björner & Wachs [10] for shellings of non-pure complexes.) Thus the complex $\Gamma = \{\emptyset\}$ has dimension -1, while every complex of dimension 0 has the form $\Gamma = \{\emptyset, \{p_1\}, \ldots, \{p_n\}\}$, where $p_1, \ldots, p_n$ are $n \geq 1$ distinct points in $\mathbb{R}^d$.

For example, the set of all faces of a polytope P forms a complex $\Gamma(P)$ of dimension $\dim(P)$ with P as its only facet, while the set of all proper faces of P, given by $\Gamma(P)\backslash\{P\}$, is a complex $\Gamma(\partial P)$ of dimension $\dim(P){-}1$, called the *boundary complex* of P. (The facets of $\Gamma(\partial P)$ are the facets of P in the sense of polytope theory [35].)

A complex is *simplicial* if all its faces are simplices, that is, convex hulls of affinely independent point sets. In the following we will also deal with cell complexes that cannot necessarily be represented by "straight" polytopal complexes. However, the class of finite regular CW complexes that we admit has the same basic combinatorial properties as polytopal complexes. We refer to Munkres [27, Sect. 38] or Bredon [11, Sect. IV.8] for the basic facts about CW complexes.

Definition 2.2 (See [8, Sect. 4.7(b)])
A CW complex is **regular** *if the closures of the* k*-cells are homeomorphic to* B^k*. It is* **finite** *if it has only finitely many cells (equivalently, if the underlying space is compact). The complex has the* **intersection property** *if it satisfies the condition* (ii) *of Definition* 2.1.

For the following, a **complex** *is the collection of all faces of a finite, regular CW complex with the intersection property. A complex is* **simplicial** *if every* k*-dimensional face has only* $k{+}1$ *vertices.*

The following definition of a shellable complex is recursive (by induction on dimension). It is slightly more restrictive than the versions used by Bruggesser & Mani [14] and by McMullen [26]. However, this version is the one that has nice combinatorial description in terms of "recursive coatom orderings" and "chainwise lexicographic (CL) shellability" in the work of Björner & Wachs [9]. For this reason, we believe that it presents the right level of generality.

Definition 2.3 (See [9], [35, Sect. 8.1])
A **shelling** *of a complex* Γ *is a linear ordering* $(F_1, F_2, \ldots, F_N)$ *of its set of facets, which is arbitrary for* $\dim(\Gamma) = 0$*, but for* $\dim(\Gamma) > 0$ *has to satisfy the following two conditions:*

(i) *the boundary complex* $\Gamma(\partial F_1)$ *has a shelling, and*

(ii) *for every* $i > 1$*, the boundary complex* $\Gamma(\partial F_i)$ *has a shelling* $(G_1, \ldots, G_M)$ *such that*

$$F_i \cap (F_1 \cup \ldots \cup F_{i-1}) \;=\; G_1 \cup \ldots \cup G_k,$$

for some $k \geq 1$*, that is, the intersection of a facet with the union of the previous facets is pure of dimension* $\dim(\Gamma) - 1$*, it is shellable, and the shelling can be extended to a complete shelling of the boundary of the facet in question.*

A complex is **shellable** *if it has a shelling.*

For polytopal complexes, the condition (i) is vacuous, since Bruggesser & Mani [14] have shown that the boundary complexes of all polytopes are shellable. However, precisely because convex polytopes are *not* extendably shellable (see Section 5) the condition "and the shelling can be extended..." is not redundant for polytopes.

Two complexes are *combinatorially equivalent* if there is a bijection between their faces that respects inclusion relations. We say that a complex is *realizable in* $\mathbb{R}^d$ if it is combinatorially equivalent to a polytopal complex in $\mathbb{R}^d$. We just note here that every d-dimensional complex all of whose faces are simplices can be realized in $\mathbb{R}^{2d+1}$, by putting its vertices on a moment curve. Not every cell complex is realizable in any $\mathbb{R}^n$: for that consider any 4-dimensional cell complex with one single 4-cell whose boundary is a 3-sphere that is not realizable as the boundary of a 4-polytope, as in [8, Sect. 5.3].

We say that a complex Γ *is a* k*-ball* if its union $\bigcup\Gamma$ is homeomorphic to the unit ball B^k in $\mathbb{R}^k$. Similarly, Γ *is a* k*-sphere* if it is homeomorphic to the unit sphere S^k in $\mathbb{R}^{k+1}$.

The following collects basic criteria for constructing and recognizing non-shellable balls and spheres. We refer to [8, Sect. 4.7] for more details, as well as for basic facts about PL balls and spheres. For our purposes, a pure d-dimensional complex is a *pseudomanifold* if every $(d-1)$-cell is contained in at most two facets. The union of the $(d-1)$-cells that are contained in only one facet form the *boundary* of the pseudomanifold.

Proposition 2.4
Let Γ *be a complex of dimension* d*.*

(i) *If* Γ *is a shellable pseudomanifold with non-empty boundary, then it is a PL d-ball (i.e., it has a subdivision that is combinatorially equivalent to a subdivision of a d-simplex).*
If Γ *is a shellable pseudomanifold without boundary, then it is a PL d-sphere (i.e., it has a subdivision that is equivalent to a subdivision of the boundary of a* $(d{+}1)$*-simplex).*

(ii) *If a d-dimensional shellable complex* Γ *is realizable in* $\mathbb{R}^d$*, then it is a PL d-ball.*

(iii) *If* Γ *is a complex with more than one facet, then a facet* F *is defined to be* **free** *if its intersection with the boundary of* Γ *is a* $(d-1)$*-ball. If* F *is free, then the complex given by the union of all other facets of* Γ *is homeomorphic to* Γ*.*

(iv) *If* Γ *is a d-ball, but has no free facet, then it is not shellable. In this case, we call* Γ **strongly non-shellable**.

(v) *If* Γ *is a non-shellable ball, and if it is simplicial or if* $d = 3$*, then* Γ *contains a strongly non-shellable ball of the same dimension.*

Proof. Part (i) is implicit in Bing [4] and explicit in Danaraj & Klee [16], see [8, Prop. 4.7.22]. With this (ii) follows as well.

For (iii) let Γ_0 be the subcomplex of Γ given by all facets other than F and their faces, let $\Gamma' \subseteq \partial F$ be the intersection of $\Gamma(F)$ with the boundary of Γ, and let Γ'' be the intersection of $\Gamma(F)$ with Γ_0. Now

F is free (by definition) if Γ' is a $(d-1)$-ball. Then also Γ'' is a $(d-1)$-ball, since the complement of a $(d-1)$-ball in the PL sphere $\Gamma(\partial F)$, is a $(d-1)$-ball. And gluing a d-ball into the boundary of a pure d-dimensional complex preserves the homeomorphism type, if the intersection is a $(d-1)$-ball.

For (iv) take a shelling of a ball. The successive unions $F_1 \cup \ldots \cup F_i$ have to be pseudomanifolds with boundary, and hence balls, by (i). Hence the last facet in any shelling has to be free.

Finally, for (v), we start with an arbitrary non-shellable ball B and perform inverse shelling steps until we get stuck. The claim is that the ball $B_0 \subseteq B$ we get stuck at, which is non-shellable by definition, must be strongly non-shellable. It has the property that for every facet F, the intersection with the union of the other facets is not a beginning of a shelling of ∂F. If $\dim(F) = 3$, or if F is a simplex, then this is equivalent with the condition that the intersection of F with the other facets is not a topological d-ball. □

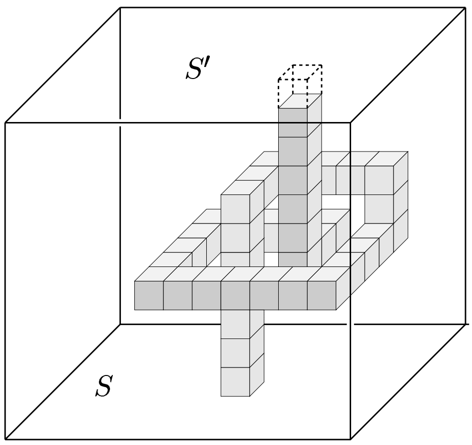
3 History

Constructions of non-shellable 3-balls abound in the topological literature. In the following, we sketch the main constructions.

3.1 Furch's "knotted hole ball."

The "knotted hole ball" may be the first example of a non-shellable 3-ball. It appears already in Furch's 1924 paper [20], but was popularized by Bing [4, p. 110]. (A different description of the same idea was given by van Kampen in 1941 [23, Footnote 2].) The construction is very important because of its great flexibility; we use it extensively in Section 5 below. Here is Furch's own description of what to do (translated freely):

> "Our starting point is a cube-shaped combination K of congruent cubes (m^3 cubes, say). Starting from one face S of the conbination we drill a channel into the interior, by removing cubes, which ends at the opposite face S' and which is embedded in the cube in a knotted fashion. If we choose m large enough then this is certainly possible. The cube that was taken away last, and whose removal has caused the breakthrough of the channel to S', is put back in, and after this we have before us the combination of cubes K', which represents the example we were after." [20, pp. 72-73]



Such a 3-ball cannot be shellable, because its 1-skeleton contains a knotted curve with all edges on the boundary, except for one edge that passes through the interior, namely one of the edges of the "plug cube." (For a variant of this argument see Lemma 5.2 below.)

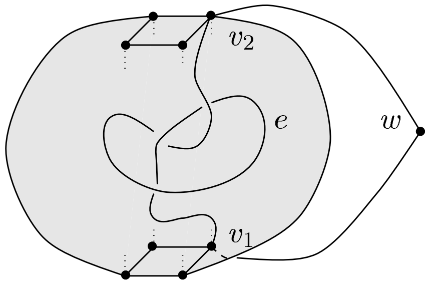
This type of "knotted hole ball" can also be constructed as a triangulation: just triangulate the complex without new vertices. In general it is not strongly non-shellable; however, with Proposition 2.4(v) one can find a strongly non-shellable subcomplex in any given example.

3.2 Non-shellable 3-spheres.

Any (simplicial) "knotted hole ball" B can be completed to a (simplicial) sphere that has a knot in a cycle with only three edges: For this one adds to B a cone over the boundary, forming $S := B \cup (\partial B * w)$ with a new vertex w. The resulting 3-sphere S contains the "knotted edge" $e \subseteq B$, which has its end points $\partial e = \{v_1, v_2\}$ in the boundary of B. Thus S also contains the edges $v_1 * w$ and $v_2 * w$, and hence a triangle

$$e \cup (v_1 * w) \cup (v_2 * w) \;=\; e \cup (\partial e * w) \;\subseteq\; B \cup (\partial B * w) \;=\; S$$

that is knotted in S with the same knot that was originally used for the construction of B. This is illustrated in our figure, where the shaded ball represents a convex homeomorphic image of the original ball B.



Lickorish [24] proved that, if the knot used here is complicated enough (for example, a connected sum of three trefoil knots), the resulting PL 3-sphere cannot be shellable. He also quotes an example of Lickorish & Martin [25] showing that a knot which is too simple (for example, a single trefoil knot) will not suffice for this.

An alternative, though similar, construction of non-shellable 3-spheres, based on sufficiently knotted complete graphs, can be found in Armentrout [1].

3.3 Newman's and Grünbaum's 3-balls.

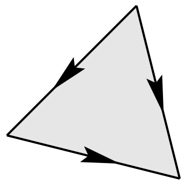
A different construction of a simplicial strongly non-shellable 3-ball was given by Newman in 1926. His construction is entirely explicit, though somewhat "wasteful": it produces (if you proceed exactly according to Newman's paper) a simplicial 3-ball with 30 vertices, all of them on the boundary, and 72 facets. From Newman's description, it is not clear whether the construction has a realization in $\mathbb{R}^3$.

Grünbaum simplified Newman's construction, and attempted to reach the smallest number of vertices possible. He arrived at a strongly non-shellable simplicial ball with 14 vertices, all of them on the boundary, and 29 facets. Unfortunately, the only account that remains is the list of facets in Danaraj & Klee [17, p. 40], without a geometric explanation, and therefore also without a known straight realization.

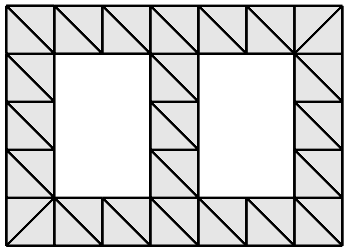
The geometric example with 10 vertices that we construct in Section 4 below is also derived by modification and simplification of Newman's construction. However, we describe it in terms of a straight geometric construction in $\mathbb{R}^3$.

3.4 Frankl's 3-ball and the "house with two rooms."

In 1931 Frankl [19] gave a third construction method that creates strongly non-shellable triangulations of 3-dimensional balls that are realized in $\mathbb{R}^3$. His idea is very beautiful and simple. One starts with a contractible, non-shellable 2-dimensional complex K in $\mathbb{R}^3$ (for example, a realization of the "dunce hat" [34] [11, p. 50] obtained by identifying all three edges of a triangle as indicated in the figure).



A tubular neighborhood of K is homeomorphic to B^3. It can be triangulated with all vertices on the boundary, and in such a way that there is no free tetrahedron. We illustrate this by a triangulation of a 2-dimensional (not contractible) region with all vertices on the boundary, but without a free triangle in the following figure.



This construction yields a 3-ball, since its boundary is an oriented 2-manifold embedded in $\mathbb{R}^3$ whose interior is contractible. The 3-ball is strongly non-shellable by Proposition 2.4(iv).

The "house with two rooms" described in Bing [4] is a variant of Frankl's construction, based on a different 2-dimensional complex that is easier to visualize (*see* [4, Figure 11]!) than the dunce hat, and for which it is also easy to see contractability ("just fill the rooms").

3.5 Rudin's non-shellable tetrahedron.

M. E. Rudin's famous non-shellable ball [29] was published in 1958. It is given by a straight, non-shellable triangulation of a tetrahedron, using 14 vertices (all of them on the boundary), and 41 facets. Rudin's example is a very interesting object, but *quite* difficult to visualize.

However, Rudin's example has an extra interesting property: its vertices can be perturbed into convex position [15, Addendum on p. 305], so we obtain the Rudin ball as a triangulation "without new vertices" of a convex 3-polytope with 14 vertices. (It is not in general possible to take a triangulation of a polytope and make it strictly convex by perturbing the boundary vertices; see Connelly & Henderson [15] and Bloch [12].)

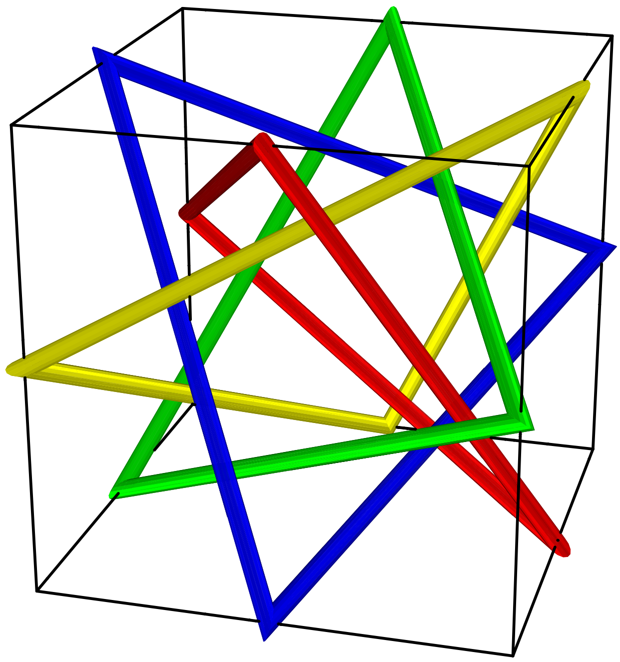
In view of the 14-vertex Grünbaum and Rudin balls, Danaraj & Klee [17, p. 40] say: "It would be interesting to know what is the minimum number of vertices, and of facets, for an unshellable 3-ball."

We believe that the construction in Section 4 below — a non-shellable simplicial 3-ball with 10 vertices and 21 facets — may be getting close to this minimum. However, there are good reasons not to conjecture that our new example is minimal. In fact, it seems that no method different from complete classification of the possible configurations can verify that examples are minimal, and this complete classification would need the enumeration of a huge number of configurations, and thus an enormous computational effort. The parameters to be checked here would be rank 4 configurations of 9 or 10 points. This is not very far outside the current range of oriented matroid technology, in view of the enumeration of uniform oriented matroids of rank 3 up to 10 points and of rank 4 up to 8 points by Bokowski & Richter-Gebert, see [13, Table 6].

Still, we are far from confirming Rudin's claim: "It can be shown that no triangulation which has less than 14 vertices has the desired property" [29, p. 90] (i.e., to be a non-shellable triangulation of a tetrahedron, possibly with all vertices on the boundary), which to my knowledge has never been substantiated in print.

3.6 The Danzer cube.

A further construction of a triangulation of a tetrahedron (or of a cube), based on a different topological obstruction, was given in [35, Example 8.9]. The basic observation for it was that any triangulation of a 3-polytope that contains the following configuration of 12 edges with all their 12 vertices on the boundary (plus some number of additional vertices and edges, triangles and tetrahedra) cannot be shellable. (The key point is that every edge in the configuration is "surrounded" by one of the triangles.) Furthermore, we argued in [35], based on the Whitehead lemma, that such triangulations (of a tetrahedron or a cube) exist; however, for this additional vertices will be neccessary.



4 A 10-vertex ball

In the following we describe a simple construction of a strongly non-shellable simplicial non-convex 3-ball B_{10} that has only 10 vertices, all of them on the boundary. (The construction uses elements from the Newman [28] and Rudin [29] balls.) It is easy to describe due to its simple coordinates for a realization in $\mathbb{R}^3$.

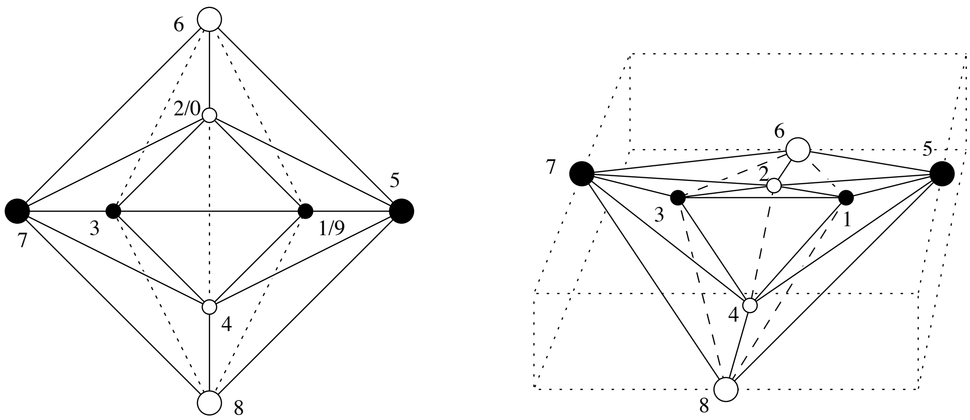
The vertex set of B_{10} is labeled $1, 2, \ldots, 8, 9, 0$. We represent it by the points in $\mathbb{R}^3$ given by the columns of the following matrix:

$$\begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccccccccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 & \scriptstyle 8 & \scriptstyle 9 & \scriptstyle 0 \end{array} \\ \begin{pmatrix} 1 & 0 & -1 & 0 & 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 2 & 0 & -2 & 0 & 1 \\ 2 & 1 & 2 & 1 & 3 & 0 & 3 & 0 & 0 & 3 \end{pmatrix} \end{array}$$

The ball B_{10} has the following list of 21 facets:

1234	1256	1569	2560	3678	4578
	2367	1629	2670	3248	4137
	3478	1249	2730	3268	4157
	4185	1489	2310		
		1859	2150		

A construction of B_{10} can be visualized as follows. We start with the center simplex 1234, and add a ring of four tetrahedra, as given by the next column of the facet list. The configuration we then have is depicted in the following figure, where the first view is seen from a point of view of very high z-coordinate, and the second is a parallel projection.



Here the vertices are indicated according to their heights, where ⬤ denotes a vertex of height 3, ● denotes height 2, ○ denotes height 1, and ◯ denotes height 0.

Then we add a new point "0" on top of point 2, and join it to the five triangles that are adjacent to 2 in the top view. This amounts to the tetrahedra in the fourth column above. Symmetrically, we add a new point "9" below point 1, and with it the five tetrahedra based on the triangles containing 1 on the lower envelope of the configuration: this is the third column in the list above.

Finally, to get a 3-ball we have to fill the space above point 4, and below point 3. This could be done in the same way by adding two new points and ten new facets, arriving at a non-shellable ball with 12 vertices and 25 facets. However, the following method is more effective: the "holes" above 4 and below 3 can be filled without new vertices, by adding the three tetrahedra each in the last two columns of the facet list.

In the resulting simplicial 3-ball, the first five tetrahedra we started with have only their vertices and some edges in the boundary, while all other tetrahedra have one triangle, the opposite vertex, and possibly some edges, in the boundary. Thus, we have a strongly non-shellable simplicial 3-ball on 10 vertices and 21 facets. □

5 Not extendable shellability

Bruggesser & Mani's "rocket flight" [14] [35, Thm. 8.11] demonstrates that (the boundary complexes of) convex polytopes are shellable; this geometric construction in fact produces a large collection of shellings for each convex polytope. In view of this, Tverberg's question [17, p. 39] whether all polytopes are *extendably shellable* is very natural: can one "get stuck" in the attempt of shelling polytopes? That is, is there, for every shellable $(d-1)$-ball Γ_0 in the boundary of a d-polytope P, a shelling of the boundary of P that first shells the ball Γ_0? The first counterexamples were presented in [35, Sect. 8.1]. The object here is to expand on the proofs in [35], and derive the same result for the classes of simple and of simplicial polytopes (as announced in [35]).

It seems to be quite difficult to show for special classes of complexes that they are extendably shellable. For example, an intriguing conjecture due to Simon [32] [35, Problem 8.24(iii)*] is that the k-skeleton of an n-dimensional simplex is extendably shellable. This was proved by Björner & Eriksson [7] for the case $k \leq 2$, and by Kalai (unpublished) for $n - k \leq 2$. It remains open in general. Similarly, for the family of d-dimensional crosspolytopes it is not at all clear that they are extendably shellable [35, Problem 8.1(iv)*]; see Hoke [21] for a recent discussion of this problem.

5.1 Most polytopes are not extendably shellable

In general, 4-dimensional polytopes are not extendably shellable: there are several different constructions and arguments given for that in [35]. Therefore we review here only the simplest and most flexible of these constructions, based on the "knotted hole ball."

The following lemma establishes that, at least for simplicial polytopes, it is sufficient to consider the 4-dimensional case. The lemma may be also true, but is less obvious, in the case of non-simplicial polytopes. (Compare to [35, Problem 8.4*].)

Lemma 5.1
If a simplical polytope is extendably shellable, then also all of its quotients (iterated vertex figures) are extendably shellable.

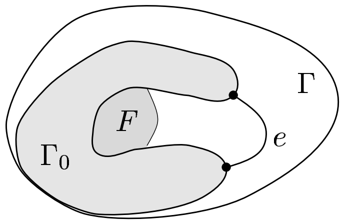
Proof. This follows from the facts that any link of a vertex in a shellable simplicial complex is shellable [35, Lemma 8.7], and that every shelling of a star of a vertex in the boundary complex of a polytope can be extended to a shelling of the whole boundary complex [26] [35, Cor. 8.13]. □

With the following theorem, we essentially establish that all "sufficiently complicated" polytopes, and thus "most d-polytopes," of dimension $d \geq 4$ are not extendably shellable. We are not trying to make the "most d-polytopes" part precise. However, we show that a 4-polytope is "sufficiently complicated" if, for example, its boundary admits a "knotted curve of facets" that is closed except for a single edge. By Lemma 5.1, a simplicial d-polytope is "sufficiently complicated" if it contains a sufficently complicated 4-face. In fact, we use the following criterion to show that a ball or sphere is not extendably shellable.

Lemma 5.2 (See Bing [4, p. 110])
Let $\Gamma_0 \subseteq \Gamma$, where Γ_0 is a shellable 3-ball, and Γ is a (shellable) 3-ball or 3-sphere. If the 1-skeleton of Γ contains a closed cycle that is knotted in Γ, and which except for one single edge $e \in \Gamma \backslash \Gamma_0$ is contained in the boundary of Γ_0, then no shelling of Γ_0 can be extended to a shelling of Γ. In particular, in this situation Γ is not extendably shellable.

Proof. We verify that whenever we extend Γ_0 by one single shelling step, which adds a facet F to Γ_0, the existence of "a knotted cycle C with only the edge e outside" is maintained. This will prove that by extending Γ_0 by legal shelling steps we will not be able to include e, and hence we cannot obtain all of Γ.

When e is not contained in F, then the endpoints of e are contained in the boundary of the ball $\Gamma_0 \cup F$. We may replace the path $C \cap \Gamma_0$ between these endpoints by a path that is also in the boundary of $\Gamma_0 \cup F$. This affords only a change of one contiguous piece of the knot C within the boundary of the 2-sphere $\partial\Gamma_0$, and hence does not change the type of the knot.



If e is contained in F, then both endpoints of e are in the 2-dimensional intersection $\Gamma_0 \cap F$. In this case we can replace the boundary part $C \cap \partial\Gamma_0$ by an edge path that stays within $F \cap \partial\Gamma_0$, while keeping the endpoints fixed. Thus we would obtain a knotted cycle in the boundary of F, which is impossible. □

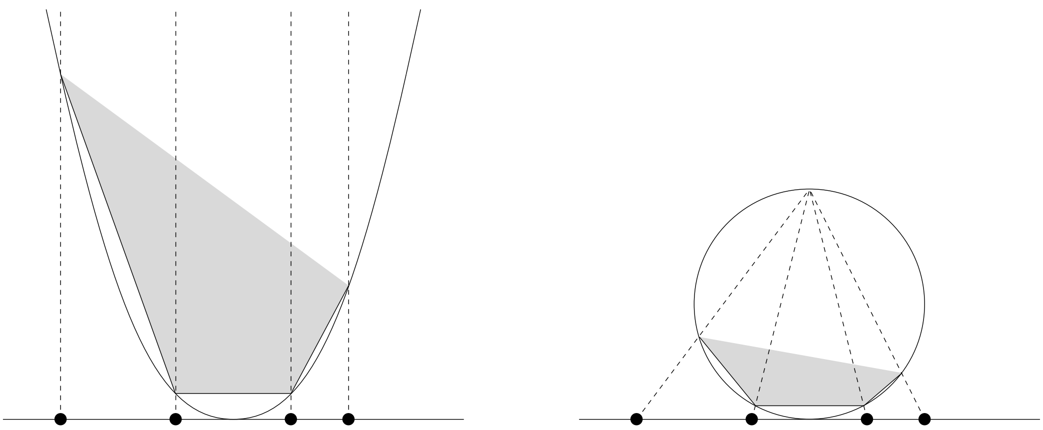
For any finite set of points $A \subseteq \mathbb{R}^d$ the *Delaunay subdivision* $\Gamma^D(A)$ is a polytopal complex that subdivides $\mathrm{conv}(A)$ and that is characterized by the following *empty sphere criterion*: For $B \subseteq A$, the polytope $\mathrm{conv}(B)$ is a face of $\Gamma^D(A)$ if and only if there is a $(d-1)$-sphere $S \subset \mathbb{R}^d$ such that $B \subseteq S$, and all the points in $A \backslash B$ lie *outside* S.

If the points in A are in general position — that is, no $d+1$ on a hyperplane, no $d+2$ on a sphere, then the subdivision is a triangulation, known as the *Delaunay triangulation* of the point set. This triangulation is of greatest importance for all of Computational Geometry [18], among other fields.

An explicit construction of the Delaunay triangulation is obtained by lifting the points to a paraboloid via the map

$$\mathbb{R}^d \longrightarrow \mathbb{R}^{d+1}, \qquad \mathbf{x} \longmapsto (\mathbf{x}, \textstyle\sum_{i=1}^d x_i^2),$$

taking the convex hull, and considering its lower faces. Equivalently (via a projective transformation that takes the paraboloid to a sphere and fixes the space (hyperplane) $\mathbb{R}^d$ of the original point configuration), one can take a stereographic projection to and from the north pole of a Riemann sphere. Both constructions are (for $d = 1$) illustrated in the following figure.



Because of this, we call a polytope P a *Delaunay polytope* if all its vertices are on a sphere. Putting the ("well-known") pieces of information [18] [8, Sect. 1.8] about Delaunay triangulations together, we obtain the following criterion. (See [35, Chap. 5] for the basic facts about Schlegel diagrams.)

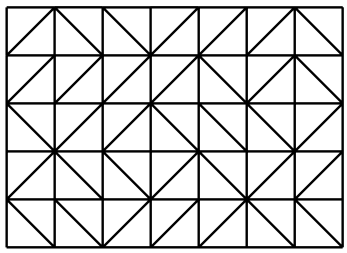
Proposition 5.3
Let $A \subseteq \mathbb{R}^d$ be a point configuration whose Delaunay subdivision is a triangulation, and whose union is

a simplex with no extra vertices on the boundary. Then the Delaunay triangulation $\Gamma^D(A)$ *is the Schlegel diagram of a simplicial Delaunay polytope.*

Delaunay triangulations are regular, and hence shellable: this is a consequence of the Bruggesser-Mani construction [14] [35, Cor. 8.14].

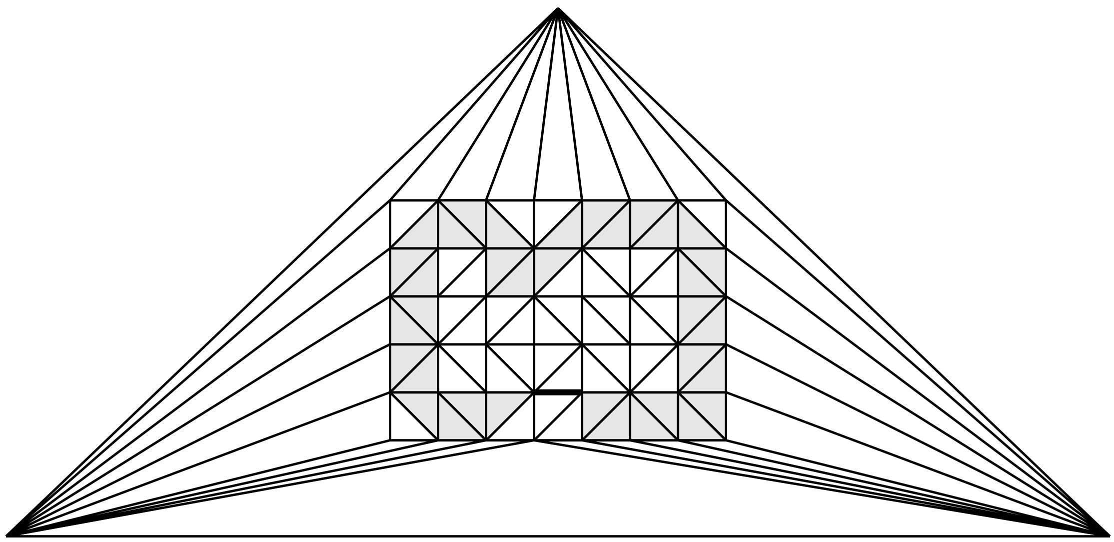
Theorem 5.4
There are simplicial Delaunay 4*-polytopes that are not extendably shellable.*

Proof. In $\mathbb{R}^3$, take a sufficiently large array of $m_1 \times m_2 \times m_3$ points, and perturb it slightly, into general position. Then the Delaunay construction yields a regular triangulation of a 3-polytope that is "approximately" a (triangulated) pile of bricks. A 2-dimensional version of this might look as follows.



Add four extra points that span a large simplex that contains all the points of the array in its interior. If the simplex is chosen large enough, then no circle spanned by points of the array contains a simplex vertex: this will have the effect that the Delaunay triangulation of the array together with the four simplex points contains the Delaunay triangulation of the array as a subcomplex. This is the Schlegel diagram of a simplicial Delaunay 4-polytope, by Proposition 5.3.

Now for Γ_0 we take a chain of facets $F_1, F_2, \dots, F_K$ such that for $1 < i \leq K$ the intersection $F_i \cap (\bigcup_{1 \leq j \leq i-1} F_j) = F_i \cap F_{i-1}$ is a facet both of F_i and of F_{i-1}. In particular, this chain is then a partial shelling. We choose it such that it follows a knotted curve and "nearly" closes it, to the extent that the first facet F_1 and the last facet F_K in the chain are connected by an edge. If m_1, m_2 and m_3 are large enough, this will certainly be possible. Our figure gives a 2-dimensional representation of the situation — the only condition that is (neccessarily) not met here is that the curve we follow should be knotted.

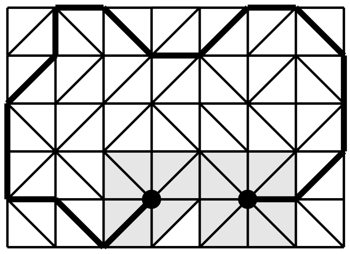


Following the knotted chain of facets is a sequence of correct shelling steps. However, this cannot be extended to a shelling of the triangulated tetrahedron, or of the corresponding 4-dimensional polytope, by Lemma 5.2. □

The polar of a simplicial polytope is simple [35, Sect. 2.5].

Theorem 5.5
There are simple 4*-polytopes that are not extendably shellable.*

Proof. Here we use the same construction as for the previous theorem, except this time we find a knotted (1-dimensional) curve, which we close to the point where the closed stars of the first and the last vertex have a facet in common. (Our figure provides, again, a 2-dimensional sketch without the knotting.)



This describes a closed, knotted path in the 1-skeleton of a simplicial Delaunay 4-polytope P. The polar P^{Δ} to P is a simple 4-polytope which has a knotted chain of adjacent facets (which is a partial shelling) such that the first and the last facet are connected by an edge. Again, Lemma 5.2 guarantees that the shelling of the polytope cannot be completed. □

5.2 Delaunay triangulations and a Beichl-Sullivan algorithm

A Delaunay triangulation can (efficiently) be computed by following a suitable shelling of the associated Delaunay polytope; see Seidel [31].

Beichl & Sullivan [3] have constructed and tested several different algorithms to compute 2- and 3-dimensional Delaunay triangulations while following a shelling, using the ("local") empty sphere criterion that we have described above. They report [2, 3, 33] that their algorithms are very fast and reliable in practice, and can very efficiently be implemented in a way that does not run into numerical difficulties (which in floating point arithmetic is a serious problem for all types of Delaunay triangulation algorithms).

However, for the versions of their algorithms that rely exclusively on *local* data, and thus follow a quite general type of shelling — these are the ones that can efficiently be parallelized — there was no guarantee that they would not get stuck. And, in fact, our proof for Theorem 5.4 yields explicit 3-dimensional point configurations on which these shelling algorithms, with an unfortunate choice of "next tetrahedron," will get stuck.

One can avoid this pitfall only by suitable *global* precautions, such as

- sorting the point configuration according to a generic linear function, and then always inserting the lexicographically first simplex available (this corresponds to "shelling according to a hyperplane sweep"), or
- following a Bruggesser-Mani shelling (this guides some of the Beichl-Sullivan implementations [3], where they chose the "next tetrahedron" to have a small μ parameter, where the minimal μ would determine the next tetrahedron for a Bruggesser-Mani shelling).

In 3 dimensions the first procedure cannot get stuck, while in the second case we have a guarantee in arbitrary dimension. However, the use of global criteria destroys the local nature of the algorithm that is needed in order to make it fast and easy to parallelize.

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