Three Problems about 4-Polytopes

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To some extent, we can claim to "understand" 3-dimensional polytopes. In fact, Steinitz' Theorem

- "the combinatorial types of 3-polytopes are given by the simple, 3-connected, planar graphs"
  (Steinitz, see Steinitz & Rademacher [12])

reduces much of the geometry of 3-polytopes to entirely combinatorial questions. Its powerful extensions answer basic questions about representing combinatorial types by actual 3-dimensional polytopes:

- "every 3-polytope can be realized with rational vertex coordinates"
  (a trivial consequence of the inductive proof for Steinitz' theorem),
- "every combinatorial type of 3-polytopes can be realized with the shape of one facet (2-face) arbitrarily prescribed"
  (a theorem obtained by subtle adaption of the proof, by Barnette & Grünbaum [2]),
- "the space of all realizations of a convex 3-polytope, up to affine equivalence, is contractible, and thus in particular connected"
  (this is what Steinitz actually proved, see [12]).

For high-dimensional polytopes, there seems to be no simple analogue to Steinitz' theorem, and analogues of all the extensions mentioned above are known to fail in sufficiently high dimensions.

In this note we describe the situation in dimension 4. Here very basic problems are open. In fact, more systematic statements (like universality theorems) are not available for polytopes in any fixed dimension d. This is due to our methods to construct and analyze high-dimensional polytopes: affine Gale diagrams and the Lawrence construction. If they are employed to transfer results about planar configurations to polytopes, the polytopes produced will be very high-dimensional, where the dimension will grow with the size of the planar configuration. (A method by Sturmfels [14] that would produce 6-polytopes is in error.)

In the following, we briefly sketch the current situation for the 4-dimensional analogues of the three extensions mentioned above. Via Schlegel diagrams, 4-polytopes can be visualized by 3-dimensional polytopal complexes. Thus there may be ways to attack the problems below by direct geometric methods, e. g. explicit constructions of Schlegel diagrams in $\mathbb{R}^3$.

However, we have to carefully distinguish 3-diagrams (polytopal complexes that "look like Schlegel diagrams") from actual Schlegel diagrams that can be lifted to 4-polytopes. All the questions we mention may (and should) be posed independently for 3-diagrams, and for 2-dimensional polytopal complexes in $\mathbb{R}^3$.

For all the basic terminology, basic constructions, examples and further references we refer to Grünbaum's fundamental volume [9], and also to [15]. In particular, see

- [9, Chap. 13] [15, Chap. 4] for Steinitz' Theorem,
- [9, Sect. 3.3] [15, Chap. 5] for Schlegel diagrams and d-diagrams,
- [9, Chap. 5] [15, Chap. 6] for Gale diagrams,
- [3] [4, Sect. 9.3] [15, Sect. 6.6] for the Lawrence construction, and
- [11] [7, Sect. 6.3] [4, Sect. 8.6] for Mnëv's Universality Theorem.

1. Prescribing the shape of a 2-face

For 4-polytopes, the shape of a facet (i.e., a 3-face) cannot be prescribed. An example with a minimal number of vertices (8 vertices, 14 facets) to that effect was constructed by Kleinschmidt [10]. Kleinschmidt's example has one octahedron facet, all other faces are simplices. An example with 8 vertices and only 9 facets (one octahedron, four square pyramids and four tetrahedra) will be given in [15, Ex. (6.12)], improving on an example with 11 facets in the preprint version.

Similarly, an example with a minimal number of facets (10 vertices, 7 facets) was independently constructed and analyzed by Barnette as a Schlegel diagram [1], and by Sturmfels via the affine Gale diagram [13]. They both studied the same example, namely the prism over a square pyramid.

However, one is tempted to ask whether "prescribing a facet" is the right high-dimensional version of the Grünbaum-Barnette Theorem. Perhaps one can at least prescribe a 2-face?

It turns out that even this is impossible for $d \geq 5$. In fact, in [15, Sect. 6.5(c)] we construct (the affine Gale diagram of the polar of) a 5-polytope (12 vertices, 10 facets) with a hexagon 2-face that cannot be arbitrarily prescribed.

This leaves us with the question:

- for every 4-dimensional polytope, can one prescribe the shape of one 2-face?

It would be very surprising if the anwer was "yes". However, Gale diagrams seem to be the wrong tool to construct counterexamples. Perhaps one can construct explicit Schlegel diagrams, in the spirit of [1]?

2. Non-rational 4-polytopes

A polytope P is *rational* if there is a combinatorially equivalent polytope P' such that all the vertices of P' have rational coordinates. Klee had asked a long time ago [9, p. 92] whether every polytope is rational.

The first non-rational polytope was an 8-polytope with 12 vertices, constructed (in terms of its Gale diagram) by Perles, see [9, p. 94] [15, Sect. 6.5(a)]. This is still the smallest known example. In fact, d-polytopes with at most $d+3$ vertices are easily represented with rational coordinates (they have 1-dimensional affine Gale diagrams). Moreover, it can probably be shown that any non-rational d-polytope with $d+4$ vertices must have $d \geq 8$, using the fact (which follows from [8]) that every planar configuration of 8 points has a rational realization.

However, what about polytopes with more vertices and facets? It would be very surprising if one could show that every 4-polytope is rational. On the other hand, no non-rational polytope with $d < 8$ is known. We formulate the problem for $d = 4$, since this is the first open case, and since it might be within the reach of direct geometric construction (via Schlegel diagrams):

- is every 4-dimensional polytope rational?

The same question is open for 3-diagrams, which are polytopal complexes in $\mathbb{R}^3$. It is even still open for 2-dimensional polyhedral complexes in $\mathbb{R}^3$ (cf. [9, p. 93]). This might be a good point to start geometric investigations.

3. Universality of realization spaces

Let P be a d-polytope P with vertex set $\{v_1, v_2, \ldots, v_n\}$, and for simplicity assume that the vertices $v_1, v_2, \ldots, v_i$ affinely span a face of dimension $i-1$, for $i \leq d+1$.

The *realization space* $\mathcal{R}(P)$ — the set of coordinatizations of P, up to affine equivalence — can then be represented by a set of all $(d \times n)$-matrices $X = (\mathbf{x}_1, \ldots, \mathbf{x}_n) = (x_{ij})$ such that the first $d+1$ columns are fixed (for example, $x_{ij} = \delta_{ij}$ for $j \leq d+1$), and such that the convex hull of the columns of X is combinatorially isomorphic to P via $\mathbf{x}_j \longmapsto v_j$.

It is not hard to see that the realization space $\mathcal{R}(P)$ is a semi-algebraic variety: a subset of $\mathbb{R}^{d\times n}$ that can be defined by polynomial equations and strict inequalities. In particular, it is a topological space with interesting structure. The realization space is connected if and only if every coordinatization of P can be "deformed" into any other coordinatization in such a way that the combinatorial type is preserved during the deformation. The *isotopy problem* for polytopes asks whether this is always the case.

Steinitz' theorem (as above) states that the realization space is contractible, and hence connected, for every 3-polytope. In contrast to this, we know that this fails badly for high-dimensional polytopes. In fact, from Mnëv's Universality Theorem together with the Lawrence construction we know that the realization spaces of polytopes can have the homotopy types of arbitrary finite simplicial complexes. This is true even if we restrict to the case of simplicial polytopes [7, Sect. 6.2]. However, the polytopes which prove this (derived from planar point configurations via the Lawrence construction) are high-dimensional, by construction.

Still, we know one example of a simplicial 4-polytope with disconnected realization space, realizing Kleinschmidt's sphere from [5] with 10 vertices and 28 facets. It has a combinatorial symmetry that cannot be realized. Mnëv observed [11] that by general principles (Smith Theory) this implies that the realization space cannot be contractible. Bokowski & Guedes de Oliveira [6] [7, Sect. 6.2] found that, indeed, it is disconnected.

One can connect several copies of this polytope to get 4-polytopes whose realization space has 2^k connected components. However, it is not proved that the realization space can be arbitrarily complicated (e.g., have the homotopy type of a circle). We certainly think that it can — but a proof method (possibly deriving this from Mnëv's Universality Theorem for point configurations) is not currently available. Thus, we ask whether 4-polytopes are universal:

- for every simplicial complex Δ, is there a 4-polytope whose realization space is homotopy equivalent to Δ?

More precisely, for every semi-algebraic variety one could ask for a polytope whose realization space is stably equivalent to it. A positive answer is not known for d-polytopes in any fixed dimension d.

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