# “What is a Complex Matroid?”

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**Abstract.** Following an ‘Ansatz’ of Björner & Ziegler [BZ], we give an axiomatic development of finite sign vector systems that we call *complex matroids*. This includes, as special cases, the sign vector systems that encode complex arrangements according to [BZ], and the complexified oriented matroids, whose complements were considered by Gel’fand & Rybnikov [GeR].

Our framework makes it possible to study complex hyperplane arrangements as entirely combinatorial objects. By comparing with the structure of *2-matroids*, which model the more general 2-arrangements introduced by Goresky & MacPherson [GoM], one can isolate the essential combinatorial meaning of a “complex structure”.

Our development features a topological representation theorem for 2-matroids and complex matroids, and the computation of the cohomology of the complement of a 2-arrangement, including its multiplicative structure in the complex case. Duality is established in the cases of complexified oriented matroids, and for realizable complex matroids. Complexified oriented matroids are shown to be matroids with coefficients in the sense of Dress & Wenzel [Dr1], but this fails in general.

**Contents.**

## 0. Introduction.

For $z \in \mathbb{C}$ define a *complex sign* $\mathbf{s}^{(1)}(z) \in \{i, j, +, -, 0\}$ according to whether $z$ imaginary part is positive or negative, or $z$ is real and positive, negative or zero:

$$\mathbf{s}^{(1)}(z) \ := \ \begin{cases} i & \text{if } \Im(z) > 0, \\ j & \text{if } \Im(z) < 0, \\ + & \text{if } \Im(z) = 0 \text{ and } z > 0, \\ - & \text{if } \Im(z) = 0 \text{ and } z < 0, \\ 0 & \text{if } z = 0. \end{cases}$$

If $\mathcal{B} = \{H_1, \ldots, H_n\}$ is an arrangement of linear complex hyperplanes in $\mathbb{C}^d$ given by $H_a = \ker(\ell_a)$, then we can associate a sign vector

$$\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}) \ := \ \big(\mathbf{s}^{(1)}(\ell_1(\mathbf{z})), \ldots, \mathbf{s}^{(1)}(\ell_n(\mathbf{z}))\big) \ \in \ \{i, j, +, -, 0\}^n$$

with every $\mathbf{z} \in \mathbb{C}^d$. Using this simple procedure described in [BZ, Sect. 1], one can encode every complex hyperplane arrangement $\mathcal{B}$ in $\mathbb{C}^d$ by a system

$$\mathcal{K} = \mathcal{K}(\mathcal{B}) \ := \ \big\{\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}) : \mathbf{z} \in \mathbb{C}^d\big\} \ \subseteq \ \{i, j, +, -, 0\}^n$$

of "complex sign vectors". This sign vector system encodes the topological structure of $\mathcal{B}$ up to homeomorphism. It satisfies properties that are quite analogous to the (covector) axioms for oriented matroids.

More generally, let $\mathcal{B}'$ be a *2-arrangement* in the sense of Goresky & MacPherson [GoM]: an arrangement of codimension-2 subspaces in $\mathbb{R}^{2d}$ such that all intersections have even codimension. After a choice of defining equations a very similar procedure [BZ, Sect. 8] associates a sign vector system $\mathcal{K}(\mathcal{B}') \subseteq \{i, j, +, -, 0\}^n$ with every 2-arrangement.

In the following, we will identify the main structural properties of the sign vector systems of complex arrangements and of 2-arrangements, and use them to develop axiom systems for *2-matroids* and *complex matroids*. This proposes an answer to the question in the title, which arises naturally from our work in [BZ], and was also posed to us by R. D. MacPherson.

The two axiom systems are shown to be complete in the following sense: every 2-matroid can be represented by a pseudo 2-arrangement, that is, by an arrangement of codimension-2 subspheres in some $S^{2d-1}$ (with an "oriented matroid frame"). This technical result allows us to reduce the theory of 2-matroids to the well-studied [BLSWZ] theory of oriented matroids. Every complex matroid satisfies an extra axiom, which is strong enough to guarantee a topological structure that resembles a complex arrangement at least up to its cohomological invariants.

The motivation for all this work is at least fourfold:

— The axiomatic framework allows us to test the quality of the combinatorial model. A successful development of the theory indicates that one has identified all the relevant combinatorial properties. Here "successful" means that the combinatorial model can be shown to capture the main geometric and topological properties of the situation. In our case, the main tests are

(i) topological representation,

(ii) duality theory, and

(iii) cohomological structure.

— The combinatorial framework can help in the study of complex arrangements, by explaining and exploiting the analogy to real arrangements and their description in terms of oriented matroids.

— The question about how well the geometric situation can be approximated by combinatorial methods is quite fundamental. Striking recent instances of this question are the problem of the "MacPhersonian" which appears in the work of Gel'fand & MacPherson on Pontrjagin classes [MPh] [GeM], and the "Generalized Baues Problem" that occurred in Billera, Kapranov & Sturmfels' [BKS] work on polyhedral subdivisions.

— Finally, the question *"What is a complex structure?"* is a fundamental one. In the axiomatic approach to 2-matroids this gets a precise framework: we can develop the combinatorial model for "even subspace arrangements" à la Goresky & MacPherson [GoM], and then isolate the combinatorial conditions that distinguish the arrangements with a complex structure from general ones.

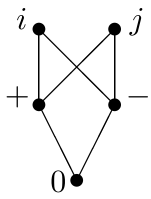
For the development of this theory, we have three different options.

1. We could do a plain axiomatic development, starting from a basic axiom system. This could be done very much **in parallel** to the theory of oriented matroids, as exposited in [BLSWZ].
2. Starting from the axioms of a 2-matroid, we can construct an oriented matroid of double rank that "subdivides" it. Then the well-developed theory of oriented matroids can be used to derive the fundamental structural properties of 2-matroids. This amounts to a development of the theory of 2-matroids **on top of** oriented matroid theory. <u>This will be done in the following</u>.
3. One could design a theory of "matroids with cellular coefficients" that covers the theory of oriented matroids as well as the theory of complex matroids (in various stratifications) and other relevant cases (e.g. the matroids with quaternionic coefficients). This development **in generality** would probably be the most powerful alternative. One step in this direction is the theory of "matroids with coefficients" of Dress & Wenzel [Dr1] [DW1,2,3], which is very much in the spirit of what is needed. However, it is (in our opinion) too closely modeled after the paradigm of rings modulo a subgroup of units, and does not cover the case of 2-matroids, see Section 6. We will not attempt to carry out this program here.

## 1. An Axiom System.

Most of the following conventions, definitions and terminology are derived from oriented matroid conventions [BLSWZ, Chap. 4], as used before in [BZ, see Def. 4.1]. We repeat them here in order to make this discussion reasonably self-contained. Also, we adapt the notation of elimination sets from [Kar], which turns out to be quite useful.

**Definition 1.1.** The set of *complex signs* is $\Sigma_2 := \{i, j, +, -, 0\}$. Its partial order is given by $i, j > +, - > 0$, that is:




Let $E$ denote a finite *ground set.* A *(complex) sign vector* is an element of $\{i, j, +, -, 0\}^E$. Sign vectors are partially ordered component-wise.

**Definition 1.2.** Let $Z, W \in \{i, j, +, -, 0\}^E$ be complex sign vectors.

(i) The *real set* of $Z$ is

$$\mathrm{Re}(Z) \ := \ \{a \in E : Z_a \in \{+, -, 0\}\},$$

the *zero set* of $Z$ is

$$\mathrm{Ze}(Z) \ := \ \{a \in E : Z_a = 0\}.$$

Hence, $\mathrm{Ze}(Z) \subseteq \mathrm{Re}(Z) \subseteq E$.

We use the notation $\underline{Z} := \{e \in E : Z_e \neq 0\} = E \backslash \mathrm{Ze}(Z)$ for the *non-zero support.*

(ii) A sign vector $X$ will be called *real* if it lies in $\{+, -, 0\}^E$, that is, if $\mathrm{Re}(X) = E$.

A sign vector $Z$ is *imaginary* if it lies in $\{i, j, 0\}^E$, that is, if $\mathrm{Re}(Z) = \mathrm{Ze}(Z)$. In this case we can write it as $Z = iY$, for a real sign vector $Y$.

(iii) The *composition* of $Z$ and $W$ is the sign vector $Z \circ W \in \{i, j, +, -, 0\}^E$ defined component-wise by

$$(Z \circ W)_a := Z_a \circ W_a = \begin{cases} W_a & \text{if } W_a > Z_a, \\ Z_a & \text{otherwise.} \end{cases}$$

(iv) The *separation set* of $Z$ and $W$ is

$$\begin{aligned} S(Z, W) \ &:= \ \{a \in E : Z_a = -W_a \neq 0\} \\ &= \ \{a \in E : (Z \circ W)_a \neq (W \circ Z)_a\}. \end{aligned}$$

(v) Every $Z \in \{i, j, +, -, 0\}^E$ can be written as $Z = X \circ iY$ for real vectors $X, Y \in \{+, -, 0\}^E$. In this case $Y$ and $Y \circ X$ are uniquely determined.

In this situation, we write $\Im(Z) := Y$ for the *imaginary part* of $Z$, and $\mathfrak{E}(Z) := i(Y \circ X)$ for the *epsilon scaling* of $Z$.

Note that all of these operations on vectors are performed component-wise, which makes them easy to handle. They should all be interpreted with respect to the $\mathbf{s}^{(1)}$-sign function

$$\mathbf{s}^{(1)} : \mathbb{C} \longrightarrow \{i, j, +, -, 0\}, \qquad x+iy \longmapsto \begin{cases} i & \text{if } y > 0, \\ j & \text{if } y < 0, \\ + & \text{if } y = 0,\ x > 0, \\ - & \text{if } y = 0,\ x < 0, \\ + & \text{if } y = x = 0. \end{cases}$$

This is also applied to vectors component-wise, so that if $\mathbf{z} \in \mathbb{C}^d$, then $Z := \mathbf{s}^{(1)}(\mathbf{z})$ records the complex signs of the components of $\mathbf{z}$. The ratio behind part (v) of Definition 1.2 is that $\mathfrak{E}(Z) = \mathbf{s}^{(1)}(e^{i\epsilon}\mathbf{z})$ for small enough $\epsilon > 0$. In particular, $\mathfrak{E}(Z)$ is always imaginary, $\mathfrak{E}(\mathfrak{E}(Z)) = \mathfrak{E}(Z)$, and $\mathrm{Ze}(\mathfrak{E}(Z)) = \mathrm{Ze}(Z)$.

**Definition 1.3.** (see [BZ, §9.4]) Let $E$ be a finite ground set. A *2-matroid* is a set $\mathcal{K} \subseteq \{i, j, +, -, 0\}^E$ of complex sign vectors that satisfies the following set of axioms.

(K0) $(00\dots0) \in \mathcal{K}$, [Zero]

(K1) $Z \in \mathcal{K}$ implies $-Z \in \mathcal{K}$, [Symmetry]

(K2) for all $Z, Z' \in \mathcal{K}$ we have $Z \circ Z' \in \mathcal{K}$, [Composition]

(K3) for all $Z', Z'' \in \mathcal{K}$ and $f \in S(Z', Z'')$, there is a $Z \in \mathcal{K}$ with
$Z_f < Z'_f$,
$Z_g = (Z' \circ Z'')_g$ for $g \notin S(Z', Z'')$, and
$Z_g \leq Z'_g$ or $Z_g \leq Z''_g$ for $g \in S(Z', Z'')$, [Elimination]

(K4) for $Z \in \mathcal{K}$, $Z_g \neq 0$ and $\sigma \in \{+, -, i, j\}$, there is a $Z' \in \mathcal{K}$ with $\mathrm{Ze}(Z') = \mathrm{Ze}(Z)$ and $Z'_g = \sigma$. [Rotation]

These axioms are analogs of the covector axioms for oriented matroids [EdM] [BLSWZ, 4.1.1]. In fact, $\mathcal{L} \subseteq \{+, -, 0\}^n$ is an oriented matroid if and only if it satisfies (K0) – (K3). In this case, $i\mathcal{L} \subseteq \{0, i, j\}^n$ and $\mathcal{L} \uplus \{(ii\dots i), (jj\dots j)\}$ also satisfy the axioms (K0) to (K3), but they are not 2-matroids. (For example, this includes the system $\{(00), (+0), (-0), (ii), (jj)\}$.) The only 'new ingredient' is the rotation axiom (K4).

We note that (K0) is a simple consequence of (K1) and (K3).

The third condition in (K3), for the case $g \in S(Z', Z'')$, was not included in [BZ, Thm. 4.3(3), 9.4(3)]. Note that it is vacuous in the oriented matroid case. We do not know whether it is implied by the other axioms. It enters the proofs for Lemmas 1.5 and 2.3, which are crucial steps of our development.

Let $\mathcal{K} \subseteq \{i, j, +, -, 0\}^E$ be a sign vector system, and let $Z', Z'' \in \mathcal{K}$ be two sign vectors and $f \in S(Z', Z'')$. We say that $Z \in \mathcal{K}$ *eliminates $f$ between $Z'$ and $Z''$* if

$$\begin{cases} Z_g < Z'_g & \text{for } g = f, \\ Z_g = (Z' \circ Z'')_g & \text{for } g \notin S(Z', Z''), \\ Z_g \leq Z'_g \text{ or } Z_g \leq Z''_g & \text{for } g \in S(Z', Z'') \backslash f. \end{cases}$$

We define the *elimination set* $I_f(Z', Z'')$ to be the set of all sign vectors that eliminate $f$ between $Z'$ and $Z''$.

With this, we can restate the 2-matroid axioms in more "shorthand" notation:

| | | |
|---|---|---|
| (K0) | $\mathbf{0} \in \mathcal{K}$, | [Zero] |
| (K1) | $\mathcal{K} = -\mathcal{K}$ | [Symmetry] |
| (K2) | $\mathcal{K} \circ \mathcal{K} \subseteq \mathcal{K}$ | [Composition] |
| (K3) | $f \in S(Z', Z'')$ implies $I_f(Z', Z'') \neq \emptyset$, | [Elimination] |
| (K4) | $\mathrm{Ze}\{Z \in \mathcal{K} : Z_g \in \{0, \sigma\}\} = \mathrm{Ze}(\mathcal{K})$ for $\sigma \neq 0$, $g \in E$. | [Rotation] |

In the next section, we will give a precise topological representation of all 2-matroids. The following should suffice as examples, for the moment. See also Example 4.3, which contains a classification of all 2-matroids on 2 elements.

**Proposition 1.4.**

*(1) For every complex arrangement, the sign vector system $\mathcal{K} := \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbb{C}^d)$, as described in the introduction, forms a 2-matroid.*

*(2) The sign vector system derived from any 2-arrangement $\mathcal{B} = \{H_1, \ldots, H_n\}$ in $\mathbb{R}^{2d}$ by choosing defining equations (as in [BZ, Ex. 8.5] [Z, Thm. 4.1]) is a 2-matroid. For this let $H_a = \{\mathbf{z} \in \mathbb{R}^{2d} : l_a(\mathbf{z}) = l'_a(\mathbf{z}) = 0\}$, where $\mathrm{rank}\{l_a, l'_a : a \in A\}$ is even for all $A \subseteq \{1, \ldots, n\}$, and set*

$$\mathcal{K} \;:=\; \big\{\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}) := \big(\mathbf{s}^{(1)}(l_1(\mathbf{z}) + i l'_1(\mathbf{z})), \ldots, \mathbf{s}^{(1)}(l_n(\mathbf{z}) + i l'_n(\mathbf{z}))\big) : \mathbf{z} \in \mathbb{R}^{2d}\big\}.$$

*(3) The sign vector system $\mathcal{K} := \mathcal{L} \circ i\mathcal{L}$ of a complexified oriented matroid is a 2-matroid, where $\mathcal{L} \subseteq \{+, -, 0\}^n$ is the covector span of an oriented matroid [GeR] [BZ, Ex. 8.6].*

*(4) The system $\mathcal{K} := \mathcal{L}' \circ i\mathcal{L}$ for $\mathcal{L}, \mathcal{L}' \subseteq \{+, -, 0\}^n$ is a 2-matroid if and only if $\mathcal{L}', \mathcal{L}$ are oriented matroids, and the underlying matroids of $\mathcal{L}'$ and $\mathcal{L}$ coincide (i.e., $\mathrm{Ze}(\mathcal{L}) = \mathrm{Ze}(\mathcal{L})$). In this case we refer to $\mathcal{K} := \mathcal{L}' \circ i\mathcal{L}$ as a quasi-complexified 2-matroid.*

**Proof.** Note that (1) is a special case of (2), and (3) is a special case of (4). The verification is easy in each of these cases: see also the the proof given for part (1) in [BZ, Thm. 4.3].

For (2), we can find $\mathbf{z} \in \mathbb{R}^{2d}$ with $\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}) = Z$ for every $Z \in \mathcal{K}$. Then we have $\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{0}) = \mathbf{0}$ for (K0), $\mathbf{s}^{(1)}_{\mathcal{B}}(-\mathbf{z}) = -\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z})$ for (K1), and $\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z} + \epsilon\mathbf{z}') = \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}) \circ \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{z}')$ for small enough $\epsilon > 0$. For (K3) we consider $\mathbf{s}^{(1)}_{\mathcal{B}}(\lambda\mathbf{z}+(1-\lambda)\mathbf{z}')$ and adjust $\lambda$ (with $0<\lambda<1$) to satisfy $\mathbf{s}^{(1)}((l_f+il'_f)(\lambda\mathbf{z}+(1-\lambda)\mathbf{z}')) < \mathbf{s}^{(1)}((l_f+il'_f)(\mathbf{z}))$. Finally, for (K4) we consider $\pm\mathbf{s}^{(1)}(\frac{1}{z_g}\mathbf{z})$ and $\pm\mathbf{s}^{(1)}(\frac{i}{z_g}\mathbf{z})$.

In the situation of (4) it is easy to see that $\mathcal{L},\mathcal{L}'$ have to be oriented matroids, see parts (ii) and (iii) of Lemma 1.5 below. For the converse, (K0), (K1) and (K2) are satisfied by construction. For (K3), consider $X'\circ iX, Y'\circ iY \in \mathcal{L}'\circ i\mathcal{L}$. If $e \in S(X,Y)$, then we take $U \in I_e(X,Y)$, to find that $(X'\circ Y')\circ iU \in I_e(X'\circ iX, Y'\circ iY)$ eliminates correctly. For $e \in S(X',Y')$ with $X_e = Y_e = 0$, we take $U' \in I_e(X',Y')$, and check that $U'\circ i(X\circ Y) \in I_e(X'\circ iX, Y'\circ iY)$ eliminates as required. Finally, (K4) requires that for every non-zero covector $X \in \mathcal{L}$ we find a non-zero covector $X' \in \mathcal{L}'$ with the same support, and conversely. Considering minimal non-zero covectors (cocircuits), this means that the underlying matroids of $\mathcal{L}$ and $\mathcal{L}'$ coincide. $\square$

In the following lemma we construct a matroid and two oriented matroids associated with a 2-matroid, generalizing the discussion of [BZ, Sect. 4].

**Lemma 1.5.** *Let $\mathcal{K} \subseteq \{i,j,+,-,0\}^E$ be a 2-matroid.*

*(i) $L := \{\mathrm{Ze}(Z) : Z \in \mathcal{K}\}$, ordered by inclusion, is a geometric lattice.*

*The corresponding matroid is denoted by $M = M(\mathcal{K})$. The rank of $\mathcal{K}$ is defined to be $r = r(\mathcal{K}) := r(M(\mathcal{K})) = r(L)$.*

*(ii) $\mathcal{L} := \{\Im(Z) : Z \in \mathcal{K}\} \subseteq \{+,-,0\}^E$ is the set of covectors of an oriented matroid $\mathcal{M}^{\mathbb{R}} = \mathcal{M}^{\mathbb{R}}(\mathcal{L})$, whose rank is denoted by $r^{\mathbb{R}} := r(\mathcal{M}^{\mathbb{R}})$.*

*(iii) $\mathcal{L}' := \mathcal{K} \cap \{+,-,0\}^E$ is the set of covectors of an oriented matroid $\mathcal{M}' = \mathcal{M}'(\mathcal{L})$, whose rank is denoted by $r' := r(\mathcal{M}')$.*

*(iv) There are strong maps of matroids $M^{\mathbb{R}} := \underline{\mathcal{M}}^{\mathbb{R}} \longrightarrow M \longrightarrow \underline{\mathcal{M}'} =: M'$. In particular, $r' \le r \le r^{\mathbb{R}}$.*

**Proof.** (i) We consider $L := \mathrm{Ze}(\mathcal{K}) \subseteq 2^E$ as a collection of subsets ('flats') of $E$. It contains $E$ by (K0) and is closed under intersection by (K2).

For every flat $A' \in L$ we have to show that the minimal flats that properly contain $A'$ partition $E\backslash A'$. Let $e \notin A'$ and let $\overline{A'\cup e}$ be the smallest flat that contains $A'\cup e$. If it is not minimal, then we find $A' \subset A'' \subset \overline{A'\cup e}$ and $f \in A''\backslash A'$ with $e \notin A''$.

Now by (K4) we can find $Z', Z'' \in \mathcal{K}$ with $\mathrm{Ze}(Z') = A'$, $\mathrm{Ze}(Z'') = A''$, and $Z'_e = -Z''_e = +$. By (K3) there is some $Z \in \mathcal{K}$ with $Z_e = 0$, $Z_f \neq 0$ and $f \in \underline{Z} \subseteq (\underline{Z'} \cup \underline{Z''})\backslash e = \underline{Z'}\backslash e$. Hence for $A := \mathrm{Ze}(Z)$ we get $e \in A$ with $A' \subset A$, $f \notin A$, contradicting the definition of $\overline{A'\cup e}$.

(ii) The axioms (K0) — (K3) for 2-matroids reduce to the covector axioms for oriented matroids [EdM] [BLSWZ, 4.1.1] under the map $Z \longmapsto \Im(Z)$.

(iii) Similarly, the axioms (K0) — (K3) for the 2-matroid $\mathcal{K}$ restrict to the oriented matroid axioms for $\mathcal{L}'$.

(iv) Let $A = \mathrm{Ze}(Z) \in L$ for some $Z \in \mathcal{K}$. By reorientation (K4) we get for every $e \in E \backslash A$ a covector $Z^{(e)} \in \mathcal{L}$ with $\mathrm{Ze}(Z^{(e)}) = \mathrm{Ze}(Z)$ and $Z^{(e)}_e = i$. Composing all these covectors $Z^{(e)}$ for $e \in E \backslash A$ we get by (K2) a covector $Z' \in \mathcal{L}$ with $\mathrm{Ze}(Z') = \mathrm{Ze}(Z)$ and $Z' \in \{i, j, 0\}^E$. For this $Z'$ we find $A = \mathrm{Ze}(Z) = \mathrm{Ze}(Z') = \mathrm{Ze}(\Im(Z')) \in \mathrm{Ze}(\mathcal{L}) = L(\mathcal{M}^{\mathbb{R}})$.

Finally, for $X \in \mathcal{L}'$ we also have $X \in \mathcal{K}$, so every flat of $M'$ is a flat of $M$. □

The following theorem characterizes quasi-complexified 2-matroids and complexified oriented matroids. Its second part extends [BZ, Prop. 5.8], whose proof is not applicable here.

**Theorem 1.6.**

*(1) The following conditions are equivalent, for every 2-matroid $\mathcal{K} \subseteq \{i, j, +, -, 0\}^E$.*

*(i)* $r = r^{\mathbb{R}}$.

*(i′)* $r = r'$.

*(ii)* $M = M^{\mathbb{R}}$, *i.e. for every* $Z \in \mathcal{K}$ *there is some* $Z' \in \mathcal{K}$ *with* $\underline{Z'} = \underline{\Im(Z)}$.

*(ii′)* $M = M'$, *i.e. for every* $Z \in \mathcal{K}$ *there is some real* $X \in \mathcal{K}$ *with* $\underline{X} = \underline{Z}$.

*(iii)* $i\Im(\mathcal{K}) \subseteq \mathcal{K}$.

*(iii′)* $\mathcal{K} = (\mathcal{K} \cap \{+, -, 0\}^E) \circ i\Im(\mathcal{K})$.

*(iv)* $\mathcal{K}$ *is a quasi-complexified 2-matroid.*

*(2) The following conditions are all equivalent, and imply the previous ones, for every 2-matroid $\mathcal{K} \subseteq \{i, j, +, -, 0\}^E$.*

*(v)* $\Im(\mathcal{K}) \subseteq \mathcal{K}$.

*(vi)* $\mathcal{K} = \Im(\mathcal{K}) \circ i\Im(\mathcal{K})$.

*(vi′)* $\mathcal{K} = (\mathcal{K} \cap \{+, -, 0\}^E) \circ i(\mathcal{K} \cap \{+, -, 0\}^E)$.

*(vii)* $\mathcal{K} \cap \{+, -, 0\}^E = \Im(\mathcal{K})$.

*(viii)* $\mathcal{K}$ *is a complexified oriented matroid.*

**Proof.** We start with part (1). (i)$\Longleftrightarrow$(ii) follows from Lemma 1.5(iv), since every rank preserving strong map of matroid on the same ground set is the identity. ¿From the same argument we get (i′)$\Longleftrightarrow$(ii′).

(iii)$\Longrightarrow$(ii) is obvious. Also, we get (iii′)$\Longrightarrow$(ii′), since the underlying matroids of $\Im(\mathcal{K})$ and of $\mathcal{K} \cap \{+, -, 0\}^E$ have to coincide by Proposition 1.4(4).

For (ii)$\Longrightarrow$(ii′) we use that every support is a union of minimal non-empty supports. Thus it suffices to consider $Z \in \mathcal{K}$ with minimal non-empty support. By (K4) we may assume that $Z$ is not imaginary. Then from (ii) we get that there is some $Z' \in \mathcal{K}$ with $\underline{Z'} = \underline{\Im(Z)} \subset \underline{Z}$, so $Z' = 0$ by minimality, and $Z$ is real.

For (ii′)$\Longrightarrow$(iii) let $Z \in \mathcal{K}$ have some $\{+, -\}$-components. By (ii′), we find a real vector $X$ with the same support as $Z$. Thus by elimination between $Z$ and $\pm X$ we find a new vector in $\mathcal{K}$ with the same imaginary components, but less $\{+, -\}$-components than $Z$. Iterating, we get $i\Im(Z) \in \mathcal{K}$.

For (iii)$\Longrightarrow$(iii$'$), let $Z = X \circ iY \in \mathcal{K}$ for $X, Y \in \{+,-,0\}^E$. ¿From (iii) we know that $iY \in \mathcal{K}$, so eliminating between $Z$ and $-iY$ we obtain a vector $Z' \in \mathcal{K}$ whose support is contained in that of $Z$, and which agrees with $Z$ in the real components of $Z$. Iterating this, we obtain a real vector $Z'' \in \mathcal{K}$ which agrees with $Z$ in the real components of $Z$, and thus $Y \circ X = Y \circ Z'' \in \mathcal{K}$.

Finally, (iii$'$)$\Longleftrightarrow$(iv) is clear.

In part (2), observe that the implications (viii)$\Longleftrightarrow$(vi)$\Longrightarrow$(v)$\Longrightarrow$(ii) are obvious.

For (v)$\Longrightarrow$(vi) we use that $M^{\mathbb{R}} \longrightarrow M'$ is a strong map by Lemma 1.5(4). Now from (v) we get that every flat of $M^{\mathbb{R}}$ is a flat of $M'$, and thus $M^{\mathbb{R}} = M'$. But now from $\Im(\mathcal{K}) \subseteq \mathcal{K} \cap \{+,-,0\}^E$ we conclude that the oriented matroids $\Im(\mathcal{K})$ and $\mathcal{K} \cap \{+,-,0\}^E$ have the same set of cocircuits, and thus they coincide.

Now (viii)$\Longrightarrow$(vi$'$)$\Longrightarrow$(vii)$\Longrightarrow$(v) is again clear, and we are done. $\square$

We will call $e \in E$ a *loop* of $\mathcal{K}$ if $Z_e = 0$ for all $Z \in \mathcal{K}$. The loops of a 2-matroid $\mathcal{K}$ coincide with the loops of its underlying matroid $M$, and with the loops of $\mathcal{M}^{\mathbb{R}}$ (by Axiom (K4)). In some cases (like Theorem 3.1) it is convenient to assume that $\mathcal{K}$ is loop-free, or that all loops have been deleted from $\mathcal{K}$ (in the obvious way).

Finally, for any sign vector $Z \in \{i,j,+,-,0\}^n$ and $A \subseteq [n] := \{1,\ldots,n\}$ we define the *reorientation* ${}_{-A}Z$ of $Z$ on $A$ by

$$({}_{-A}Z)_a \;=\; \begin{cases} -Z_a & \text{if } a \in A, \\ Z_a & \text{otherwise.} \end{cases}$$

Similarly, we define the reorientation of any sign vector system $\mathcal{K} \subseteq \{i,j,+,-,0\}^n$ by ${}_{-A}\mathcal{K} := \{{}_{-A}Z : Z \in \mathcal{K}\}$. Clearly ${}_{-A}\mathcal{K}$ is a 2-matroid if and only if $\mathcal{K}$ is one.

**Remark 1.7.** The reader may observe that the main results of our theory, including Lemma 2.4 and Theorem 3.5 below, can be generalized to 2-matroids that do not necessarily satisfy (K4). Such 2-matroids can, besides loops, contain elements that are *real* or *imaginary* in the sense that $Z_e \in \{+,-,0\}$ resp. $Z_e \in \{i,j,0\}$ for all $Z \in \mathcal{K}$. This covers the cases of oriented matroids $\mathcal{L}$ (for which every element is real) and of systems of the form $i\mathcal{L}$, for which all elements are imaginary. We will not pursue this extra generality. Let us only note that Lemma 1.5(i) becomes false without (K4).

## 2. Extension Lemma.

The goal of this section is to prove the Extension Lemma 2.4. This quite innocent-looking lemma is the key step in our reduction of 2-matroids to oriented matroids. We start with three simple preparatory lemmas. For simplicity, we write $W \in [Z', Z'']$ if for every $e \in E$ we have $W_e \leq Z'_e$ or $W_e \leq Z''_e$, or both. This is satisfied if $W \leq Z'$, or if $W \leq Z''$, or if $W \in I_e(Z', Z'')$ for some $e \in S(Z', Z'')$.

**Lemma 2.1.** *Assume that $\mathcal{K} \subseteq \{i, j, +, -, 0\}^E$ satisfies* (K2). *Then*
*(i)* $S(Z', Z'') = S(Z' \circ Z'', Z'' \circ Z')$, *and*
*(ii)* $I_f(Z', Z'') = I_f(Z' \circ Z'', Z'' \circ Z')$ *for* $f \in S(Z', Z'')$.

This is immediate from the definitions. It implies that the axiom (K3) need only be checked for pairs $Z', Z''$ with $\mathrm{Ze}(Z') = \mathrm{Ze}(Z'')$ and $\mathrm{Re}(Z') = \mathrm{Re}(Z'')$.

**Lemma 2.2.** *Let $W, Z', Z'' \in \mathcal{K}$, with $W \in [Z', Z'']$ and $f \in S(Z', Z'')$.*
*(i) If* $W_f < Z'_f$, *then* $f \in S(Z', W \circ Z'')$, *and* $I_f(Z', W \circ Z'') \subseteq I_f(Z', Z'')$.
*(ii) If* $W_f = Z'_f$, *then* $f \in S(W \circ Z'', Z'')$, *and* $I_f(W \circ Z'', Z'') \subseteq I_f(Z', Z'')$.
*(iii) If* $W_f < Z'_f$ *and* $Z \in I_f(Z', Z'')$, *then* $W \leq W \circ Z \in I_f(Z', Z'')$.

**Proof.** The proofs for (i) and (ii) are easy, since we can argue component-wise. For this note that by replacing $Z'' \longrightarrow W \circ Z''$, resp. $Z' \longrightarrow W \circ Z''$, we get smaller separation sets: $S(Z', W \circ Z'') \subseteq S(Z', Z'')$, resp. $S(W \circ Z'', Z'') \subseteq S(Z', Z'')$. Thus the stronger condition (i.e., if $g$ is not in the separation set) has to be verified on a smaller set of elements $g$.

Similarly, (iii) easily follows component-wise. We will do this proof, to demonstrate the general pattern.

$Z'_f = -Z''_f \neq 0$, so $Z_f < Z'_f$, where $W_f < Z'_f$. Thus $W_f \circ Z_f < Z'_f$.
If $Z'_g = Z''_g$, then $Z_g = Z'_g$, where $W_g \leq Z'_g$. Thus $W_g \circ Z_g \leq Z'_g$.
If $Z'_g = -Z''_g \neq 0$, then $Z_g$ and $W_g$ are $\leq Z'_g$ or $\leq Z''_g$. Thus $W_g \circ Z_g \leq Z'_g$ or $\leq Z''_g$. □

**Lemma 2.3.** *Assuming* (K2), *the axiom* (K3) *is equivalent to*

| | | |
|---|---|---|
| (K3′) | $S(Z', Z'') \neq \emptyset$ *implies* $\bigcup_{g \in S(Z', Z'')} I_g(Z', Z'') \neq \emptyset$. | [Elimination] |

**Proof.** We use induction on $|S(Z', Z'')|$: by Lemma 2.1 we can assume that we are given $Z', Z'' \in \mathcal{K}$ of the same support and $f \in S(Z', Z'')$. By (K3′) there is $Z \in I_g(Z', Z'')$ for some $g \in S(Z', Z'')$. If $Z_f < Z'_f$, then we are done. If $Z_f = Z'_f$ then we can replace $Z'$ by $Z \circ Z''$, and be done by induction and Lemma 2.2(ii). The case $Z_f = Z''_f$ is handled symmetrically. □

Now we go for the 'big step'. Figure 2.1 below is supposed to give the intuition for what is actually happening: we interpret the situation with respect to a real hyperplane

$H_e^{\mathbb{R}} \cong \{Z \in \mathcal{K} : Z_e \in \{+,-,0\}\}$ and to a flat $H_e \cong \{Z \in \mathcal{K} : Z_e = 0\} \subseteq H_e^{\mathbb{R}}$ of real codimension 2. The 'new' hyperplane $H_{e^!}^{\mathbb{R}}$ is supposed to be inserted in such a way that $H_e = H_{e^!}^{\mathbb{R}} \cap H_e^{\mathbb{R}}$, where $H_{e^!}^{\mathbb{R}}$ arises from $H_e^{\mathbb{R}}$ by a small rotation in clock-wise order. In Figure 2.1, each of the 9 cells of the subdivision of the plane by $H_e$ and $H_{e^!}$ are labeled by the pair $(Z_e, Z_{e^!})$.

More algebraically, this corresponds to the insertion of a second copy $\ker(\ell_{e^!})$ of the complex hyperplane $\ker(\ell_e)$, which is given by $\ell_{e^!}(\mathbf{z}) := e^{i\epsilon}\ell(\mathbf{z})$, for small enough $\epsilon > 0$.

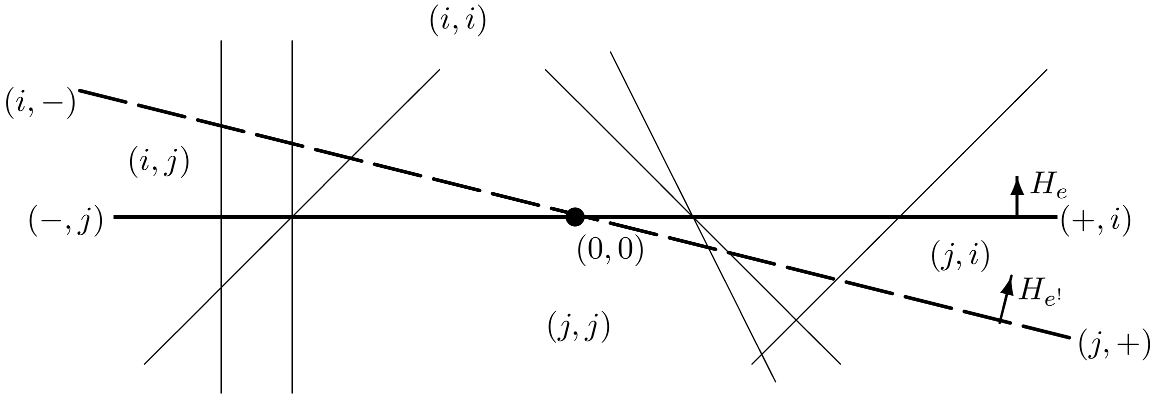



**Figure 2.1:** Sketch for the construction of $\mathcal{K} \uplus e^!$ in Lemma 2.4.

Having this picture in mind, we write down the set of sign vectors of the extended arrangement in terms of the original one, and verify that the extension again satisfies the axioms.

**Extension Lemma 2.4.** *Let $\mathcal{K} \subseteq \{i,j,+,-,0\}^E$ be a 2-matroid, and $e \in E$. Then*

$$\mathcal{K}^! \;:=\; \mathcal{K}_1^! \uplus \mathcal{K}_2^! \uplus \mathcal{K}_3^! \;\subseteq\; \{i,j,+,-,0\}^{E \uplus e^!}$$

*is a 2-matroid on the extended ground set $E \uplus \{e^!\}$, where*

$$\begin{aligned} \mathcal{K}_1^! &:= \big\{ \pm (Z,i) : Z \in \mathcal{K},\ Z_e \in \{i,+\}\big\} \\ \mathcal{K}_2^! &:= \big\{ \pm (Z,0) : Z \in \mathcal{K},\ Z_e = 0\big\} \\ \mathcal{K}_3^! &:= \big\{ \pm (Z,i), \pm(Z,+) : W, Z \in \mathcal{K},\ W \leq Z,\ W_e = +, Z_e = j\big\}. \end{aligned}$$

Note that in this Lemma, $\mathcal{K}_1^! \uplus \mathcal{K}_2^! = \{(Z, \mathfrak{C}(Z_e)) : Z \in \mathcal{K}\}$.

**Proof.** We assume that the axioms (K0) — (K4) hold for $\mathcal{K}$, and using this we verify "(K0)$^!$ — (K4)$^!$", where (K$i$)$^!$ denotes the validity of (K$i$) for $\mathcal{K}^!$. Here <u>(K0)$^!$</u> and <u>(K1)$^!$</u> are clear by construction.

**(K2)$^!$:** Let $(Z,\tau),(Z',\tau') \in \mathcal{K}^!$. Using (K1)$^!$, we may assume that $\tau \in \{0,+,i\}$. We distinguish three cases:

- If $(Z,i) \in \mathcal{K}^!_1$, then $\tau\circ\tau' = i$. We get that $(Z\circ Z',i) \in \mathcal{K}^!_1$ unless $Z_e = +$ and $Z'_e = j = (Z\circ Z')_e$. In this case we get $(Z\circ Z',i) \in \mathcal{K}^!_3$.
- If $(Z,0) \in \mathcal{K}^!_2$, then $\tau\circ\tau' = \tau'$, $Z_e = 0$, $(Z\circ Z')_e = Z'_e$. Furthermore, if $W \le Z'$, then $Z\circ W \le Z\circ Z'$ with $(Z\circ W)_e = W_e$, so $(Z',\tau') \in \mathcal{K}^!_i$ implies $(Z\circ Z',\tau\circ\tau') \in \mathcal{K}^!_i$ for $i = 1,2,3$.
- If $(Z,\tau) \in \mathcal{K}^!_3$, then $W \le Z \le Z\circ Z'$ with $W_e = +$, $(Z\circ Z')_e = Z_e = j$, $\tau \in \{+,i\}$ and $\tau\circ\tau' \in \{+,i,j\}$. In this case we conclude that $(Z\circ Z',+),(Z\circ Z',i) \in \mathcal{K}^!_3$ and $(Z\circ Z',j) \in \mathcal{K}^!_1$, so all bases are covered.

**(K3)$^!$:** Using (K2)$^!$ and Lemma 2.3, it suffices to verify (K3$'$)$^!$.

For this, let $(Z',\tau'),(Z'',\tau'') \in \mathcal{K}^!$ and $f \in S((Z',\tau'),(Z'',\tau''))$. With (K2)$^!$ and Lemma 2.1, we can assume that $(Z',\tau')$ and $(Z'',\tau'')$ have the same support, so $Z'_g = \pm Z''_g$ for all $g \in E$, and $\tau' = \pm\tau''$. We will distinguish three main cases.

To reduce the number of subcases we have to consider, we make the following observations. First, using (K1)$^!$, we may assume that $Z'_e \in \{0,+,i\}$. Also, if $Z'_e = Z''_e$ and $\tau'_e = -\tau''_e$ (in Case II below), then we may assume that $\tau'_e \in \{0,+,i\}$, using the symmetry between $(Z',\tau')$ and $(Z'',\tau'')$. Similarly, if $Z'_e = -Z''_e$ and $\tau'_e = \tau''_e$ (in Case I), then we may assume that $\tau'_e \in \{0,+,i\}$, using the symmetry together with (K1)$^!$. Also, by construction we find that $(Z'_e,\tau'),(Z''_e,\tau'') \in \{\pm(i,i),\pm(+,i),(0,0),\pm(j,i),\pm(j,+)\}$. In the following we only consider those non-trivial subcases that are not ruled out by these assumptions we can make.

**Case I**: $e \in S(Z',Z'')$, that is, $Z'_e = -Z''_e \neq 0$.

**I.1:** $Z'_e = -Z''_e = +$. This implies $\tau' = -\tau'' = i$. Choose some $Z \in I_e(Z',Z'')$. Then $Z_e = 0$, and we can use $(Z,0) \in \mathcal{K}^!_2$.

**I.2:** $Z'_e = -Z''_e = i$. Then $\tau' \in \{i,j,-\}$, $\tau'' \in \{i,j,+\}$, and there are three subcases left to consider.

- Let $\tau' = \tau'' = i$. By construction, we get $W'' < Z''$ with $W''_e = +$.

  Now $W''_e < Z''_e$, so by Lemma 2.2(i)(iii) we can find $Z \in I_e(W''\circ Z',Z'') \subseteq I_e(Z',Z'')$ with $W'' \le Z$, which implies $Z_e \in \{+,i,j\}$. For $Z_e \in \{+,i\}$ we get $(Z,i) \in \mathcal{K}^!_1$, and for $Z_e = j$ we get $(Z,i) \in \mathcal{K}^!_3$.
- For $\tau' = -\tau'' = -$ we get $W' < Z'$ and $W'' < Z''$ with $W'_e = -$, $W''_e = +$.

  Now $W'_e < Z'_e$, thus by Lemma 2.2(i)(iii) we can choose a $U' \in I_e(Z',W'\circ Z'') \subseteq I_e(Z',Z'')$ that satisfies $W' \le U'$, and thus $- = W'_e \le U'_e < Z'_e = i$, that is, $U'_e = -$. Symmetrically, we get $U'' \in I_e(Z',Z'')$ with $U''_e = +$. Now we choose $Z \in I_e(U',U'')$. This $Z$ satisfies $Z \in I_e(Z',Z'')$ (check component-wise), and $Z_e = 0$, so $(Z,0) \in \mathcal{K}^!_2$ meets all conditions.
- For $\tau' = -\tau'' \in \{i,j\}$, let $Z \in I_e(Z',Z'')$. If $Z_e = 0$, then we have $(Z,0) \in \mathcal{K}^!_2$, for $Z_e = +$ we use $(Z,i) \in \mathcal{K}^!_1$, and for $Z_e = -$ we use $(Z,j) \in \mathcal{K}^!_1$.

**Case II**: $e \notin S(Z', Z'') \neq \emptyset$. In this case we choose some $f \in S(Z', Z'')$, so we have $Z'_f = -Z''_f \neq 0$, but $Z'_e = Z''_e$.

**II.1:** $Z'_e = Z''_e = 0$. Take $Z \in I_f(Z', Z'')$, then $(Z, 0) \in \mathcal{K}^!_2$ satisfies (K3)$^!$.

**II.2:** $Z'_e = Z''_e = +$. Then $\tau' = \tau'' = i$. Taking $Z \in I_f(Z', Z'')$, we get $Z_e = Z'_e = +$, so $(Z, i) \in \mathcal{K}^!_1$ eliminates as required.

**II.3:** $Z'_e = Z''_e = i$. We are left with two subcases:

- For $\tau' = \pm\tau'' = i$ we take $Z \in I_f(Z', Z'')$, where $Z_e = i$. From this we get $(Z, i) \in \mathcal{K}^!_1$, which satisfies (K3)$^!$.
- $\tau' = \tau'' \in \{-, j\}$. In this case there are $W' < Z'$, $W'' < Z''$ in $\mathcal{K}$ with $W'_e = W''_e = -$. Now we could have $W'_f < Z'_f$, in which case we can choose $Z \in I_f(Z', W' \circ Z'') \subseteq I_f(Z', Z'')$ with $W' \leq Z$, by Lemma 2.2(i)(iii). This implies that $(Z, -), (Z, j) \in \mathcal{K}^!_3$, and we are done. Analogously we proceed if $W''_f < Z''_f$.

  The last possibility is that $W'_f = Z'_f$ and $W''_f = Z''_f$. In this case we choose $Z \in I_f(Z', Z'')$ and $W \in I_f(W', W'')$. Now by Lemma 2.2(iii) $W \leq W \circ Z \in I_f(Z', Z'')$, which satisfies $W_e = W'_e = -$ and $(W \circ Z)_e = Z_e = Z'_e = i$, so $(W \circ Z, \tau') \in \mathcal{K}^!_3$ solves the problem.

**Case III**: $S(Z', Z'') = \emptyset$, that is, $Z' = Z''$ and $f = e^!$.

$(Z', +), (Z', -) \in \mathcal{K}^!$ is impossible. If $(Z', i), (Z', j) \in \mathcal{K}^!$, then $Z'_e \in \{i, j\}$, and by (K1)$^!$ we may assume $Z'_e = j$. But this implies $(Z', +) \in \mathcal{K}^!_3$, and we are done.

**$\underline{\text{(K4)}^!}$:** We consider $(Z, \tau) \in \mathcal{K}^!$ with $(Z, \tau)_g \neq 0$, and $\sigma \neq 0$. By (K1)$^!$, we can assume that $\tau, \sigma \in \{+, i\}$.

We first treat the case $g = e^!$. For $(Z, +) \in \mathcal{K}^!$ we always have $(Z, i) \in \mathcal{K}^!$. Similarly, for $(Z, i) \in \mathcal{K}^!_3$ we have $(Z, +) \in \mathcal{K}^!_3$. For $(Z, i) \in \mathcal{K}^!_1$ we get $Z_e \in \{+, i\}$. With (K4) we find $U, W \in \mathcal{K}$ with $\underline{U} = \underline{W} = \underline{Z}$ and $U_e = +$, $W_e = j$. Thus we get $(U \circ W, +) \in \mathcal{K}^!_3$ with $\underline{U \circ W} = \underline{Z}$, as required.

Finally, assume $g \in E$: for this let $(Z, \tau) \in \mathcal{K}^!$ with $Z_g \neq 0$, and let $Z' \in \mathcal{K}$ have $\underline{Z} = \underline{Z'}$ and $Z'_g = \sigma \neq 0$. Now if $(Z, \tau) \in \mathcal{K}^!_2$, then we conclude $Z_e = Z'_e = 0$ and thus $(Z', 0) \in \mathcal{K}^!_2$ satisfies the conditions. Otherwise, if $(Z, \tau) \in \mathcal{K}^!_1 \cup \mathcal{K}^!_3$, we get that $Z_e \neq 0$, and thus $Z'_e \neq 0$. This means that we can either use $(Z', i) \in \mathcal{K}^!_1$, if $Z'_e \in \{+, i\}$, or we can use $(Z', j) \in \mathcal{K}^!_1$, if $Z'_e \in \{-, j\}$. □

Note that if $e$ is a loop in $\mathcal{K}$, then both $e$ and $e^!$ are loops in $\mathcal{K}^!$. Otherwise the matroid $M^! := M(\mathcal{K}^!)$ is obtained by inserting $e^!$ into $M$ as an element that is parallel to $e$, while $\{e, e^!\}$ is independent in the oriented matroid $\mathcal{M}^{\mathbb{R}}(\mathcal{K}^!)$.

## 3. Reduction and Representation Theorem.

At this stage of our work it is easy to construct an oriented matroid that 'represents' a given 2-matroid, although the precise analysis of the geometric situation will need some care. Definition 3.1 contains the basic construction, and we will assume its notation for the whole section. The exclamation mark will consistently be used to denote the basic doubling operation, with $F! := F \uplus \{a{+}n : a \in F\}$ for subsets $F \subseteq [n] := \{1, \ldots, n\}$: it will never be used to denote factorials or excitement.

**Definition 3.1.** Let $\mathcal{K} \subseteq \{i, j, +, -, 0\}^n$ be a loop-free 2-matroid of rank $r$ on $[n]$. Let $L := \mathrm{Ze}(\mathcal{K})$ be the geometric lattice of flats associated with $\mathcal{K}$ by Lemma 1.5(i), and $r = r(\mathcal{K}) := r(L)$ its rank.

¿From this, construct

$$\mathcal{K}! \;:=\; ((...((\mathcal{K} \uplus 1^!) \uplus 2^!) \uplus \ldots) \uplus n^!) \;\subseteq\; \{i, j, +, -, 0\}^{2n},$$

the *doubled 2-matroid* on $\{1, \ldots, n, 1^!, \ldots, n^!\} \cong [2n]$ obtained from $\mathcal{K}$ by $n$ successive single element extensions according to Lemma 2.4. The lattice of flats of $\mathcal{K}!$ will be denoted by $L! := \mathrm{Ze}(\mathcal{K}!)$.

The *oriented matroid frame* of $\mathcal{K}$ is the set of real sign vectors

$$\mathcal{L}! \;:=\; \Im(\mathcal{K}!) \;=\; \{\Im(Z) : Z \in \mathcal{K}!\} \;\subseteq\; \{+, -, 0\}^{2n},$$

which is the covector set of an oriented matroid by Lemma 1.5(ii).

The constructions in this section depend on a linear ordering of the ground set, which we prescribe by identifying the ground set with $[n] = \{1, \ldots, n\}$. For example, it is not clear that the single element extensions of Definition 3.1 commute (they don't). Similarly, for the construction of minors, which we need below, we assume tacitly that the elements of a deletion or contraction keep the induced linear order (even if they are relabeled as $\{1, \ldots, m\}$).

**Lemma 3.2.**

*(i) The map $\Im : \mathcal{K}! \longrightarrow \{+, -, 0\}^{2n}$ is injective.*

*(ii) The flats of $\mathcal{K}!$ are given by $L! = \{F! : F \in L\}$: so we get a canonical isomorphism $L \cong L!$.*

*(iii) The flats of the underlying matroid of $\mathcal{L}!$ are given by $\mathrm{Ze}(\mathcal{L}!)$. We have $F! \in \mathrm{Ze}(\mathcal{L}!)$ if and only if $F \in L$.*

**Proof.** (i) The map $\Im : \mathcal{K}! \longrightarrow \{+, -, 0\}^{2n}$ is injective, since its inverse is given by

$$\Im^{-1} : \mathcal{L}! \longrightarrow \mathcal{K}!, \qquad (X, Y) \longmapsto (Y \circ iX, (-X) \circ iY).$$

For this observe that $\Im(Z_e) = 0$ implies $Z_e = \Im(Z_{e^!})$, and $\Im(Z_{e^!}) = 0$ implies $Z_{e^!} = -\Im(Z_e)$, from the construction of Lemma 2.4.

(ii) Since the extensions of Lemma 2.4 add new elements that are parallel to old ones, we get $L \cong L!$ canonically, where $L! = \{F! : F \subseteq [n]\}$ is the geometric lattice of $\mathcal{K}!$.

(iii) For every $F \in L$ there is some imaginary sign vector $iX \in \mathcal{K}$ with $\mathrm{Ze}(iX) = F$. We conclude $(iX, iX) \in \mathcal{K}!$, hence $(X, X) \in \mathcal{L}!$ and thus $F! \in \mathrm{Ze}(\mathcal{L}!)$. For the converse let $F! \in \mathrm{Ze}(\mathcal{L}!)$ and take $(X, Y) \in \mathcal{L}!$ with $\mathrm{Ze}(X) = \mathrm{Ze}(Y) = F$. Then $\Im^{-1}(X, Y) = (Y \circ iX, (-X) \circ iY) = (iX, iY) \in \mathcal{K}!$, which implies $iX \in \mathcal{K}$ and thus $F \in L$. $\square$

The definition of minors of 2-matroids $\mathcal{K} \subseteq \{i, j, +, -, 0\}^n$ follows the model (and the geometric intuition) of oriented matroids, see [BLSWZ, Lemma 4.1.8]. We denote the deletion of $F$ from a sign vector $Z$ by $Z \backslash F = (Z_f : f \notin F)$, and the restriction to the entries in $F$ by $Z|_F = (Z_f : f \in F)$. With this, for $F \subseteq [n]$ we define the *deletion* of $F$ from $\mathcal{K}$ by

$$\mathcal{K} \backslash F \ := \ \{Z \backslash F : Z \in \mathcal{K}\},$$

and the *contraction* of $\mathcal{K}$ by $F$ as

$$\mathcal{K}/F \ := \ \{Z \backslash F : Z \in \mathcal{K}, Z|_F = 0\}.$$

Note that if $\mathcal{K}$ is a 2-matroid, then so is any deletion or contraction. The two operations commute; a *minor* is defined as any deletion of a contraction (or contraction of a deletion) of $\mathcal{K}$.

**Lemma 3.3.**

*(i)* $\mathcal{K}!/F! \ = \ (\mathcal{K}/F)!$

$\mathcal{K}! \backslash F! \ = \ (\mathcal{K} \backslash F)!$

*(ii)* $\Im(\mathcal{K}!/F!) \ = \Im(\mathcal{K}!)/F!$

$\Im(\mathcal{K}! \backslash F!) \ = \Im(\mathcal{K}!) \backslash F!$

**Proof.** For (i), we can use induction on $|F|$, and thus only consider the case $F = \{f\}$. But then this amounts to saying that contraction and deletion commute with the !-extensions. Specifically, we need four statements, which are easily checked from the construction of $\mathcal{K}' \uplus e^!$, for a 2-matroid $\mathcal{K}'$:

- $(\mathcal{K}' \uplus e^!)/f \ = \ (\mathcal{K}'/f) \uplus e^! \quad$ for $f \notin \{e, e^!\}$,
- $(\mathcal{K}' \uplus e^!) \backslash f \ = \ (\mathcal{K}' \backslash f) \uplus e^! \quad$ for $f \notin \{e, e^!\}$,
- $(\mathcal{K}' \uplus e^!) \backslash \{e, e^!\} \ = \ \mathcal{K}' \backslash e$,
- $(\mathcal{K}' \uplus e^!)/\{e, e^!\} \ = \ \mathcal{K}'/e$.

The most subtle case occurs for '$\subseteq$' in the second statement. If $(Z \backslash f, \tau) \in (\mathcal{K}' \backslash f)^!_3$ with $Z_e = j$, then there exists some $W \backslash f < Z \backslash f$ with $W_e = +$. If $W < Z$ does not hold in this case, then we replace $Z$ by $W \circ Z$, and are done.

For part (ii), we get the first equality from the proof of Lemma 3.2(i), the second one is trivial. $\square$

Note that there are no complicated isomorphisms involved in Lemma 3.3, it just states an equality of two sign vector systems (complex for part (i), real for part (ii)).

However, let us mention that one has to be careful with handling the !-construction: it certainly depends on the order of the elements, and the single element extensions that produce it cannot in general be interchanged.

**Lemma 3.4.** *Let $\mathcal{K}$ be a 2-matroid of rank 1. Then $\mathcal{L}!$ is an oriented matroid of rank 2.*

**Proof.** We may assume that $\mathcal{K}$ has no loops, so that $\mathrm{Ze}(\mathcal{K}) = \{\emptyset, [n]\}$ and $\mathcal{K}\backslash\mathbf{0} \subseteq \{i, j, +, -\}^n$.

Claim 1: for all $a \in [n]$ there is a unique $Z \in \mathcal{K}$ with $Z_a = +$. In fact, existence of $Z$ follows from (K4). If there was a second one, $Z'$ say, we would conclude $\mathbf{0} \in I_a(Z, -Z')$. Thus for all $b \in [n]$ we have either $Z_b = Z'_b = 0$ or $b \in S(Z, -Z')$, which shows $Z = Z'$.

Claim 2: for all $a \in [n]$ there is a unique $Z \in \mathcal{K}!$ with $Z_a = +$. For this we only have to show that uniqueness cannot be lost under the extensions of Lemma 2.4. For this assume $Z_a = \pm$, $Z_e = j$ and there is some $W < Z$ with $W_e = +$. But this implies either $W_a = Z_a$, contradicting Claim 1, or $W_a = 0$, contradicting that we have rank 1.

Claim 3: for all $a \in [n]$ there is a unique $(X, Y) \in \mathcal{L}!$ with $X_a = 0$, $Y_a = +$. This we get from Claim 2, using the formula for $\Im^{-1}$ in the proof for Lemma 3.2(i).

Claim 4: $r(\mathcal{L}!) = 2$. Here $r(\mathcal{L}!) \leq 2$ follows from the fact that $r(\mathcal{L}!/a) = 1$ for $a \in [n]$, by Claim 3. The rest follows from the fact that $\{a, a{+}n\}$ is independent in $\mathcal{L}$ for $a \in [n]$, by construction. $\square$

The main goal of the rest of this section is to explain the geometry of $\mathcal{K}!$ and $\mathcal{L}!$. For this we rely on

- the topological representation of oriented matroids by 1-pseudoarrangements, according to the Topological Representation Theorem [FL] [EdM] [BLSWZ, Chap. 5]. Here we use the prefix '1-' to indicate that we are talking of subspheres of codimension 1 in the sphere $S^{r(\mathcal{L})-1}$.
- the concept of *2-pseudoarrangements* $\mathcal{B}$, as defined by [BZ, Def. 8.3]: this is an arrangement of codimension 2 subspheres $\{S_1, \ldots, S_n\}$ in $S^{2r-1}$ with the property that every intersection of some of them has an even codimension. Furthermore one requires the existence of a *signed real frame*, that is, of a 1-pseudoarrangement $\{T_1, \ldots, T_n, U_1, \ldots, U_n\}$ such that every codimension 2 pseudosphere can be written as an intersection of two codimension 1 pseudospheres, $S_a = T_a \cap U_a$.
- the $\mathbf{s}^{(1)}$*-stratification* of a 2-arrangement $\mathcal{B}$ with respect to a real frame [BZ, Def. 8.6], which is the cell decomposition of $S^{2r-1}$ given by all the intersections of spheres in

$\{S_1, \ldots, S_n\}$ and spheres in $\{T_1, \ldots, T_n\}$: this is constructed as a map

$$\mathbf{s}^{(1)}_{\mathcal{B}} : S^{2d-1} \quad \longrightarrow \quad \{i, j, +, -, 0\}^n$$

which is (for $1 \le a \le n$, $\mathbf{x} \in S^{2d-1}$) defined by:

$$\left(\mathbf{s}^{(1)}_{\mathcal{B}}(\mathbf{x})\right)_a = \begin{cases} i, & \text{if } \mathbf{x} \in T_a^+, \\ j, & \text{if } \mathbf{x} \in T_a^-, \\ +, & \text{if } \mathbf{x} \in T_a \cap U_a^+, \\ -, & \text{if } \mathbf{x} \in T_a \cap U_a^-, \\ 0, & \text{if } \mathbf{x} \in T_a \cap U_a. \end{cases}$$

This cell decomposition, the $\mathbf{s}^{(1)}$*-stratification of* $\mathcal{B}$, is a regular CW decomposition of $S^{2r-1}$ by [BZ, Theorem 8.10]. Thus it is completely encoded in the associated complex sign vector system $\mathcal{K}^{(1)}_{\mathcal{B}} := \mathbf{s}^{(1)}_{\mathcal{B}}(S^{2r-1})$, with its natural partial order.

– Similarly, the $\mathbf{s}^{(2)}$*-stratification*

$$\mathbf{s}^{(2)}_{\mathcal{B}} : S^{2d-1} \quad \longrightarrow \quad \{+, -, 0\}^{2n}$$

is defined (for $1 \le a \le 2n$, $\mathbf{x} \in S^{2d-1}$) by:

$$\left(\mathbf{s}^{(2)}_{\mathcal{B}}(\mathbf{x})\right)_a = \begin{cases} +, & \text{if } 1 \le a \le n \text{ and } \mathbf{x} \in T_a^+, \\ -, & \text{if } 1 \le a \le n \text{ and } \mathbf{x} \in T_a^-, \\ 0, & \text{if } 1 \le a \le n \text{ and } \mathbf{x} \in T_a, \\ +, & \text{if } n+1 \le a \le 2n \text{ and } \mathbf{x} \in U_{a-n}^+, \\ -, & \text{if } n+1 \le a \le 2n \text{ and } \mathbf{x} \in U_{a-n}^-, \\ 0, & \text{if } n+1 \le a \le 2n \text{ and } \mathbf{x} \in U_{a-n}. \end{cases}$$

Its sign vector system $\mathcal{K}^{(2)}_{\mathcal{B}} := \mathbf{s}^{(2)}_{\mathcal{B}}(S^{2r-1}) \subseteq \{+, -, 0\}^{2n}$ is itself an oriented matroid of rank $2r$.

The following theorem contains both the reduction of 2-matroids to oriented matroids, and the topological representation theorem for 2-matroids. We will for simplicity only consider loop-free 2-matroids.

**Reduction and Representation Theorem 3.5.**
*(1) Let $\mathcal{K} \subseteq \{i, j, +, -, 0\}^n$ be a loop-free 2-matroid of rank $r$ on $[n]$, let $\mathcal{K}!$ be the corresponding doubled 2-matroid, and $\mathcal{L}!$ the oriented matroid frame of $\mathcal{K}!$.*

*If $\{T_1, \ldots, T_n, U_1, \ldots, U_n\}$ is a 1-pseudoarrangement in $S^{2r-1}$ that represents $\mathcal{L}!$, and $S_a := T_a \cap U_a$, then*

*$\mathcal{B} := \{S_1, \ldots, S_n\}$ is a 2-pseudoarrangement,*

*$\mathcal{F} := \{T_1, \ldots, T_n, U_1, \ldots, U_n\}$ is a signed real frame for $\mathcal{B}$,*

*$\mathcal{K} = \mathcal{K}^{(1)}_{\mathcal{B}}$ is the face poset of the $\mathbf{s}^{(1)}$-stratification of $\mathcal{B}$, and*

*$\mathcal{L}! = \mathcal{K}^{(2)}_{\mathcal{B}}$ is the face poset of the $\mathbf{s}^{(2)}$-stratification of $\mathcal{B}$ associated with the frame $\mathcal{F}$.*

*(2) Conversely, if $\mathcal{B} = \{S_1, \ldots, S_n\}$ is a 2-pseudoarrangement with a signed real frame $\mathcal{F} = \{T_1, \ldots, T_n, U_1, \ldots, U_n\}$, then its $\mathbf{s}^{(1)}$ face poset $\mathcal{K}^{(1)}_{\mathcal{B}}$ (with respect to the frame $\mathcal{K}$) is a 2-matroid.*

*Let $\mathcal{L}!$ be the oriented matroid frame for $\mathcal{K}^{(1)}_{\mathcal{B}}$ and let $\mathcal{F}' = \{T'_1, \ldots, T'_n, U'_1, \ldots, U'_n\}$ be a 1-pseudoarrangement in $S^{2r-1}$ that represents $\mathcal{L}!$. Then there is a homeomorphism $S^{2r-1} \longrightarrow S^{2r-1}$ that identifies $S_a = T_a \cap U_a$ with $S'_a := T'_a \cap U'_a$, and $T_i$ with $T'_i$ (but not in general $U_i$ with $U'_i$). In particular, $\mathcal{F}$ and $\mathcal{F}'$ represent the same 2-arrangement.*

**Proof.** Here we rely on the work done in [BZ, Sect. 8], in particular the fact that the $\mathbf{s}^{(1)}$-stratification of a 2-arrangement yields a regular cell complex, by [BZ, Theorem 8.10]. With this, the main point that is left to prove for (1) is that $\mathcal{B}$ is a 2-arrangement, that is, that each $F \subseteq [n]$ the set $F! = F \uplus \{a+n : a \in F\}$ has even rank in the underlying matroid $\underline{\mathcal{L}!}$, and rank 2 if $F = \{a\}$. The key to this is

$$(*) \qquad r^{\mathbb{R}}!(F!) \;=\; 2{\cdot}r(F) \qquad \text{for all} \;\; F \subseteq [n],$$

where $r$ denotes the rank function of $L$, while $r^{\mathbb{R}}!$ is the rank function of the underlying matroid of $\mathcal{L}!$.

Now $(*)$ is certainly correct for $F = \emptyset$, where both ranks vanish. We use induction on $r(F)$. Consider flats $F \lessdot F'$ in $L$, that is, $F'$ contains $F$, but with no flat between. This means that $M/F\backslash([n]\backslash F')$ has rank 1. We have to verify $r^{\mathbb{R}}!(F'!) - r^{\mathbb{R}}!(F!) = 2$.

Using (K2)+(K4) we can find imaginary vectors $iY < iY'$ in $\mathcal{K}$ with $\mathrm{Ze}(Y) = F$ and $\mathrm{Ze}(Y') = F'$. From the construction of $\mathcal{K}!$ and $\mathcal{L}!$ we now get that $(Y', Y') < (Y, Y)$ are vectors in $\mathcal{L}!$, with $\mathrm{Ze}(Y, Y) = F!$, $\mathrm{Ze}(Y', Y') = F'!$. Since the zero map on oriented matroids is order inverting and cover preserving [BLSWZ, Thm. 4.1.14(ii)], this is equivalent to showing that the oriented matroid $\mathcal{L}!/F!\backslash([n]\backslash F')!$ has rank 2. By Lemma 3.3, we get that this is $\Im\big((\mathcal{K}/F\backslash([n]\backslash F'))!\big)$. Thus we are done by Lemma 3.4.

For part (2), the key result is that $\mathcal{K}^{(1)}_{\mathcal{B}}$ is a 2-matroid: this is proved in [BZ, Prop. 8.12]. The rest now follows from the description of the geometry in [BZ, Sect. 8]. □

Theorem 3.5 allows us to develop the whole theory of 2-matroids 'on top of' the theory of oriented matroids. As examples, we list some results that can be lifted to 2-matroids by Theorem 3.5.

**Corollary 3.6.** *Let $\mathcal{K}$ be a 2-matroid of rank $r = r(M)$, whose oriented matroids have ranks $r^{\mathbb{R}} = r(\mathcal{M}^{\mathbb{R}})$ and $r' = r(\mathcal{M}')$.*

*Then $0 \le r' \le r \le r^{\mathbb{R}} \le 2r$ with $r' + r^{\mathbb{R}} = 2r$, where $r' = r = r^{\mathbb{R}}$ holds if and only if $\mathcal{K}$ is a quasi-complexified oriented matroid.*

**Proof.** $r' \le r \le r^{\mathbb{R}}$ was proved in Lemma 1.5. For $r^{\mathbb{R}} \le 2r$ we observe that $\mathcal{L}!$ is an oriented matroid extension of $\mathcal{M}^{\mathbb{R}}$, with $r^{\mathbb{R}} = r(\mathcal{M}^{\mathbb{R}}) \le r(\mathcal{L}!) = 2r$.

Now consider a topological representation of $\mathcal{K}$. Then $\mathcal{M}^{\mathbb{R}}$ is represented by the 1-pseudoarrangement $\{T_1, \ldots, T_n\}$ in $S^{2r-1}$. Thus $r^{\mathbb{R}}$ is the codimension of $\mathbf{T} := \bigcap_{a=1}^{n} T_a$

in $S^{2r-1}$. At the same time, $\mathbf{T}$ consists of all the cells with real sign vectors, and thus $\mathcal{M}'$ is represented by the 1-pseudoarrangement $\{\mathbf{T}\cap S_1,\ldots,\mathbf{T}\cap S_1\}$ in $\mathbf{T}$. Thus

$$r^{\mathbb{R}} \;=\; (2r-1)-\dim(\mathbf{T}) \;=\; (2r-1)-(r'-1) \;=\; 2r-r'.$$

The case of equality $r'=r=r^{\mathbb{R}}$ is characterized in Theorem 1.6. □

**Corollary 3.7.** *For every 2-matroid $\mathcal{K}$ of rank $r$, the poset $(\mathcal{K},\leq)$ is the face poset of a $PL$ CW-sphere $S^{2r-1}$.*

**Proof.** This follows from the $PL$ version of the Topological Representation Theorem [FL] for oriented matroids, due to Edmonds & Mandel [EdM] [BLSWZ, Chap. 5]. □

Let us now identify some interesting subposets in 2-matroids, following [BZ, Sect. 3]. By Theorem 3.5, $\mathcal{K}$ is the face poset of a $(2r-1)$-sphere. Similarly,

$$\mathcal{K}_{link} := \{Z\in\mathcal{K} : Z_a = 0 \text{ for some } a\in[n]\},$$

the *link poset*, is the face poset of a cell decomposition of $\bigcup\mathcal{B}$. Thus we get that the order complex $\Delta(\mathcal{K}_{link}\backslash 0)$ is homeomorphic to $\bigcup\mathcal{B}$. Similarly, with the yoga of [BZ, Sect. 3], we see that the poset $\mathcal{K}$ determines the *complement* space $S^{2r-1}\backslash\bigcup\mathcal{B}$ up to homeomorphism. Setting

$$\mathcal{K}_{comp} := \mathcal{K}\backslash\mathcal{K}_{link},$$

we find that $0\uplus(\mathcal{K}_{comp})^{op}$ is the face poset of a CW-complex $\Gamma_{comp}$ that is homotopy equivalent to (in fact, a deformation retract of) the complement $S^{2r-1}\backslash\bigcup\mathcal{B}$. Here $\Delta(\mathcal{K}_{comp})$ is the barycentric subdivision of $\Gamma_{comp}$.

The following collects some basic topological results. For this, recall that a *circuit* of $M$ on $[n]$ is a minimal dependent subset $C\subseteq[n]$, corresponding to a minimal subset of $\mathcal{B}$ for which the codimension of the intersection is less than twice the cardinality. A *broken circuit* is a set $C\backslash\min(C)$ for a circuit $C$. With this we define the *broken circuit complex* as the set system

$$\mathbf{BC}(M) = \{A\subseteq[n] : A \text{ does not contain a broken circuit}\}.$$

Finally, the *characteristic polynomial* of $M$ is

$$\chi(t) \;=\; \sum_{A\in\mathbf{BC}(M)} (-1)^{|A|}\, t^{r-|A|}.$$

**Corollary 3.8.** *Let $\mathcal{K}$ be a 2-matroid, with underlying matroid $M$ of rank $r$.*

*(1) The link complex $\Delta(\mathcal{K}_{link}\backslash 0)$ of a 2-arrangement is a disjoint union of circles, if $r=2$, and for $r>2$ it is homotopy equivalent to a wedge of spheres:*

$$\Delta(\mathcal{K}_{link}\backslash 0) \;\simeq\; \bigvee_{A\in\mathbf{BC}(M)\backslash\emptyset} S^{2r-2-|A|}.$$

*(2) The complement $\Delta(\mathcal{K}_{comp})$ is stably homotopy equivalent to a wedge of spheres,*

$$\Delta(\mathcal{K}_{comp}) \quad \sim \quad \bigvee_{A \in \mathbf{BC}(M)\setminus\emptyset} S^{|A|}.$$

*The cohomology algebra $\mathrm{H}^*(\Delta(\mathcal{K}_{comp});\mathbb{Z})$ of the complement has a presentation of the form*

$$0 \quad \longrightarrow \quad I \quad \longrightarrow \quad \Lambda^*\mathbb{Z}^n \quad \xrightarrow{\pi} \quad \mathrm{H}^*(\Delta(\mathcal{K}_{comp});\mathbb{Z}) \quad \longrightarrow \quad 0.$$

*Here the relation ideal $I$ is generated by relations of the form*

$$\sum_{i=0}^{k} \epsilon_i \; \mathbf{e}_{a_0} \wedge \ldots \wedge \widehat{\mathbf{e}_{a_i}} \wedge \ldots \wedge \mathbf{e}_{a_k}$$

*for circuits $A = \{a_0, \ldots, a_k\} \subseteq [n]$ and $\epsilon_i \in \{+1, -1\}$, and $\{\pi(\mathbf{e}_A) : A \in \mathbf{BC}(M)\}$ is a $\mathbb{Z}$-basis for $\mathrm{H}^*(\Delta(\mathcal{K}_{comp});\mathbb{Z})$.*
*In particular, $\mathrm{H}^*(\Delta(\mathcal{K}_{comp});\mathbb{Z})$ is free over $\mathbb{Z}$, generated in dimension 1, and has the Poincaré polynomial $(-t)^r\chi(-1/t)$.*

**Proof.** With Theorem 3.5, this are immediate consequences of [BZ, Thm. 6.6] (for $r \neq 3$) and [ZŽ, Cor. 3.3] for part (1), and of [ZŽ, Thm. 3.4] and [BZ, Cor. 7.3] for part (2). □

## 4. Complex matroids.

As we have seen, every 2-arrangement gives rise to a 2-matroid. However, 2-matroids are really more general than complex arrangements: for example, they have a completely different representation theory [GM, p. 257], and they have structurally different cohomology algebras [Z]. Therefore, the need arises to distinguish a class of complex matroids among the more general 2-matroids.

**Definition 4.1.** A *complex matroid* is a 2-matroid that satisfies the additional axiom (K5):

| | |
|---|---|
| (K5) | If $iX \in \mathcal{K}\backslash\mathbf{0}$ with $X \in \{+,-,0\}^E$, then there is $X \circ iY \in \mathcal{K}$ for some $Y < X$, $Y \in \{+,-,0\}^n$. [Complex structure] |

To get a feeling for what this axiom means geometrically, we will first prove that it holds for complex arrangements and for complexified oriented matroids. Then we will discuss examples, including a complete discussion of the case of rank 1.

**Proposition 4.2.**

*(i) Let $\mathcal{B} = \{\ker(\ell_1), \ldots, \ker(\ell_n)\}$ be a complex arrangement in $\mathbb{C}^d$. Then $\mathcal{K} := \mathbf{s}^{(1)}_{\mathcal{B}}(\mathbb{C}^d)$ is a complex matroid.*

*(ii) Let $\mathcal{L} \subseteq \{+,-,0\}^n$ be an oriented matroid. Then $\mathcal{K} := \mathcal{L} \circ i\mathcal{L}$ is a complex matroid.*

**Proof.** For (ii), we can simply choose $Y = 0$. For (i), take $\mathbf{z} \in \mathbb{C}^d$ with $Z^0 = iX = \mathbf{s}^{(1)}(\mathbf{z})$, and consider $Z^t := \mathbf{s}^{(1)}(e^{-it}\mathbf{z})$ for real $t \in \mathbb{R}$. The $\mathbf{s}^{(1)}$-function is semi-continuous, thus the $\mathbf{s}^{(1)}$-strata are open in their linear hull [BZ, Sect. 1]. Since $Z^0$ is imaginary, we conclude $Z^0 \circ Z^t = Z^0$ for all $t$. Thus $\{t \in \mathbb{R} : Z^t = Z^0\}$ is an open subset of $\mathbb{R}$, and $t_0 := \min\{t > 0 : Z^t \neq Z^0\}$ exists.

Now $Z^t_a = \mathbf{s}^{(1)}(\ell_a(e^{-it}\mathbf{z})) = \mathbf{s}^{(1)}(e^{-it}\ell_a(\mathbf{z}))$, from which we see that $\mathfrak{E}(Z^{t_0}) = Z^t$ for $0 \leq t < t_0$, which proves the claim. □

In the situation of Proposition 4.2(i), we call

- the sequence of complex linear forms $\ell_1, \ldots, \ell_n \in (\mathbb{C}^d)^*$, or, equivalently,
- the complex vector space $V := \{(\ell_1(\mathbf{z}), \ldots, \ell_n(\mathbf{z})) : \mathbf{z} \in \mathbb{C}^d\} \subseteq \mathbb{C}^n$

a *complex realization* of $\mathcal{K} = \mathbf{s}^{(1)}(V)$. For the equivalence, see [BZ, proof of Thm. 2.5].

Thus, a 2-matroid is *complex realizable* (and thus, in particular, it is a complex matroid) if there exists a complex vector subspace $V \subseteq \mathbb{C}^n$ with $\mathbf{s}^{(1)}(V) = \mathcal{K}$.

More generally, we say that a 2-matroid is *realizable* if there is a real vector subspace of $\mathbb{C}^n$ such that $\mathbf{s}^{(1)}(V) = \mathcal{K}$. In this case we necessarily have that the dimension

$$\dim_{\mathbb{R}}(V \cap \{\mathbf{z} \in \mathbb{C}^n : z_a = 0 \text{ for } a \in A\})$$

is even for all $A \subseteq [n]$. We note that the representation theory of 2-matroids and complex matroids is hardly studied at all, see also Problem 7.4.

**Example 4.3.** The following gives a complete list of the 2-matroids on two elements.

First, we have the 2-matroids with two loops, $\mathcal{K} = \{00\}$, or with exactly one loop and one coloop, $\mathcal{K} = \{00, 0+, 0-, 0i, 0j\}$ and $\mathcal{K} = \{00, +0, -0, i0, j0\}$ or with two coloops, $\mathcal{K} = \{i, j, +, -, 0\}^2$. All these 2-matroids correspond to complexified real arrangements, and are trivially complex.

The interesting case is when $\mathcal{K}$ has no loops or coloops, so the underlying matroid is $M = U_{1,2}$. By Lemma 3.4, we find four cases in this situation (up to reorientation, i.e., reversing all signs on the second element), see Figure 4.1. In the first two, the associated real matroid $M^{\mathbb{R}}$ has rank 1 as well, in the second two it has rank 2. Also, we can see that two of these 2-matroids are complex, two are not.

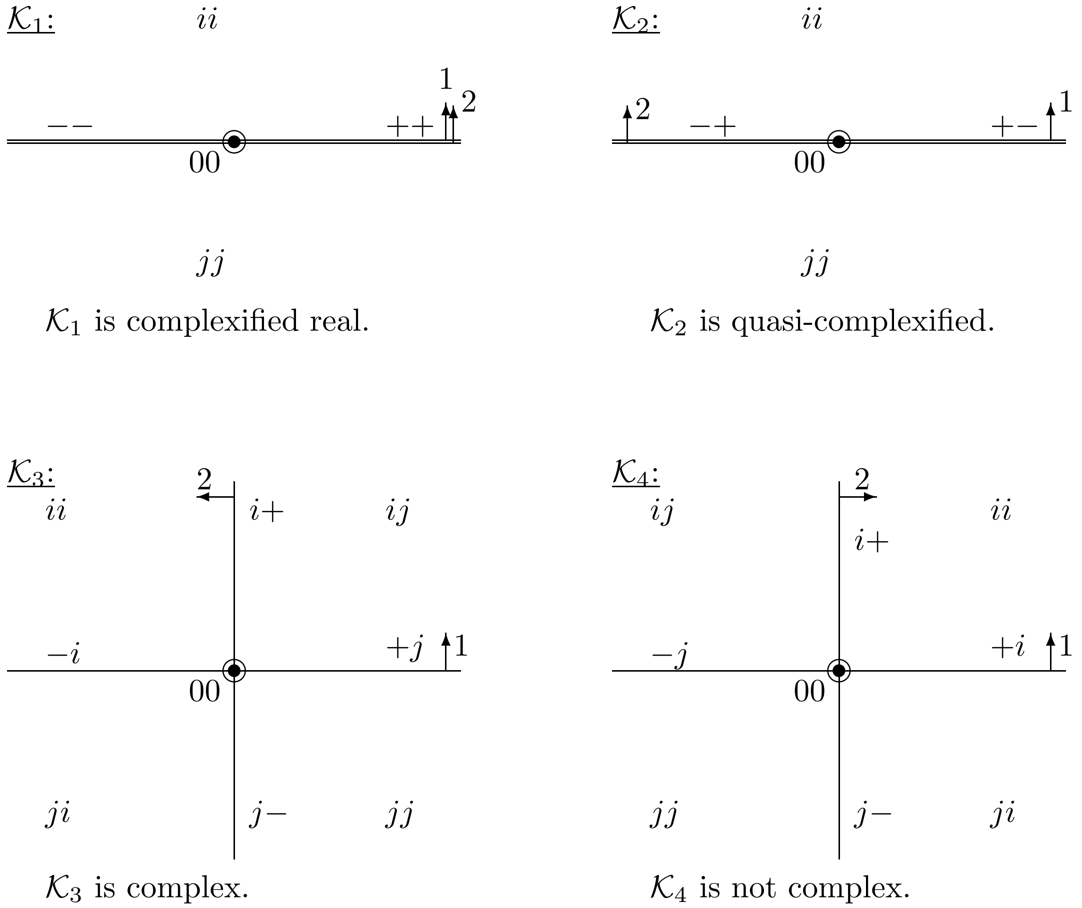



**Figure 4.1:** 2-matroids with underlying matroid $U_{1,2}$.

At this point, let us mention two alternative axioms for complex structure:

| | |
|---|---|
| (K5$'$) | If $iX \in \mathcal{K}\backslash 0$, with $X \in \{+,-,0\}^E$, then there is $(-X)\circ iY \in \mathcal{K}$ for some $Y < X$, $Y \in \{+,-,0\}^n$. |
| (K5$''$) | If $X\circ iY \in \mathcal{K}$ with $X, Y \in \{+,-,0\}^E$ and $Y \le X$, then $iX \in \mathcal{K}$. |

Note that axiom (K5$''$), which we may abbreviate by $\mathfrak{E}(\mathcal{K}) \subseteq \mathcal{K}$, is satisfied on $\mathcal{K}_4$, although this is not a complex matroid. Thus (K5$''$) does not imply (K5) for 2-matroids. We think, however, that

$$\text{(K5)} \iff \text{(K5}'\text{)} \implies \text{(K5}''\text{)}.$$

In fact, this would follow from Conjecture 4.8 below. In the following development, we will not use (K5$'$) and (K5$''$), and show that (K5) itself is good enough to ensure complex structure in rank 1, and to control the cohomological structure.

**Proposition 4.4.** *Let $\mathcal{K} \subseteq \{i,j,+,-,0\}^n$ be a 2-matroid on $[n]$ of rank 1 without loops. Then $\mathcal{K}$ can be realized over $\mathbb{C}$ by*

$$\ell_a(z) \;=\; \begin{cases} z & \text{for } a = 1, \\ \alpha_a z & \text{if the restriction } \mathcal{K}|_{\{1,a\}} \text{ is complex,} \\ \alpha_a \overline{z} & \text{otherwise,} \end{cases}$$

*for suitable $\alpha_a \in \mathbb{C}$, $2 \le a \le n$.*

**Proof.** This we get from Lemma 3.4/Theorem 3.5 and the fact that every oriented matroid of rank 2 is realizable, together with Example 4.3. □

For the four 2-matroids in Figure 4.1, we have, for example, the following coordinatizations:

$\mathcal{K}_1$: $\ell_1(z) = z$, $\ell_2(z) = z$.
$\mathcal{K}_2$: $\ell_1(z) = z$, $\ell_2(z) = -\overline{z}$.
$\mathcal{K}_3$: $\ell_1(z) = z$, $\ell_2(z) = -iz$.
$\mathcal{K}_4$: $\ell_1(z) = z$, $\ell_2(z) = i\overline{z}$.

**Lemma 4.5.** *Let $\mathcal{K} \subseteq \{i,j,+,-,0\}^n$ be a 2-matroid. If $\mathcal{K}$ is a complex matroid, then so is every minor of $\mathcal{K}$.*

**Proof.** This is trivial for a contraction $\mathcal{K}/e \cong \{Z \in \mathcal{K} : Z_e = 0\}$. Thus we consider $\mathcal{K}\backslash e = \{Z\backslash e : Z \in \mathcal{K}\}$, and take some $Z \in \mathcal{K}$ such that $Z\backslash e$ is imaginary. By (K2) we may assume $Z_e \in \{0,+,i\}$.

If $Z_e = 0$, so that $Z$ is of the form $Z = (iY, 0)$, then we are done by (K5). If $Z = (iY,+)$, then from the 2-matroid axioms we derive that also $(iY,j) \in \mathcal{K}$. Now by (K5) we either get directly that (K5) holds for $Z\backslash e$, or we conclude that $(iY,-) \in \mathcal{K}$, in which case we can eliminate $e$ between $(iY,+)$ and $(iY,-)$, to get $(iY,0) \in \mathcal{K}$. Finally, if $Z = (iY, i)$, then we can directly apply (K5), and we are either done, or we conclude $(iY, +) \in \mathcal{K}$, so we are reduced to the previous case. $\square$

**Corollary 4.7.** *Every complex matroid of rank* 1 *is complex realizable. Furthermore, there is a natural bijection between the complex matroids of rank at most* 1 *and the oriented matroids of rank at most* 2 *on the same ground set.*

**Conjecture 4.8.** *If* $\mathcal{K}$ *is a* 2*-matroid of rank* $r$*, and* $\mathcal{K}\backslash A$ *is complex for every deletion of rank* 1 *and of corank* 1*, then* $\mathcal{K}$ *is complex.*

This conjecture is an analogue to the “corank 2 theorems” about oriented matroids, see [LV] [BLSWZ, Sect. 3.6] [DW3], since by Lemma 3.4 the 2-matroids of rank 1 have an oriented matroid frame of rank 2.

Some intuition for these questions comes from the example $\mathcal{B}'$ of [Z]: it leads to non-complex 2-matroids of corank 2 over the matroid $U_{2,4}$. Here every 1-point deletion has corank 1. The point is that every deletion can be made complex, by possibly conjugating elements, but there is no complex structure globally.

**Theorem 4.9.** *The cohomology algebra* $\widetilde{\mathrm{H}}^*(\Delta(\mathcal{K}_{link}\backslash 0); \mathbb{Z})$ *of any complex matroid* $\mathcal{K}$ *is isomorphic to the Orlik-Solomon algebra* $OS(M)$ *of the underlying matroid, that is, for a complex matroid one can choose* $\epsilon = (-1)^i$ *in the presentation of Corollary 3.8(2).*

**Proof.** We need that the cohomology classes constructed in [BZ, Cor. 7.3] satisfy the Orlik-Solomon relations. This is true in complex arrangements, and it is preserved under deletion of elements that are not involved in the relations.

Thus we only have to show that every complex matroid of corank at most 1 is complex realizable. One way to see this is to combine Corollary 4.7 with the Duality Theorem 5.3. $\square$

We note that this contains both the result for complex arrangements due to Orlik & Solomon [OrS], and the result for complexified oriented matroids, due to Gel’fand & Rybnikov [GeR] [Ry].

## 5. Duality.

In this Section, we will start the work on a duality theory for complex matroids. For this, we denote by $A^T$ the transpose and by $\overline{A}$ the conjugate of a matrix (or vector) $A$. We call two vectors $\mathbf{z}, \mathbf{w} \in \mathbb{C}^d$ *orthogonal* if $\langle \mathbf{z}, \mathbf{w} \rangle := \mathbf{z}^T \overline{\mathbf{w}} = 0$. This can be used to define that $Z, W \in \{i, j, +, -, 0\}^n$ are *orthogonal*, denoted by $Z \perp W$, if there exist $\mathbf{z}, \mathbf{w} \in \mathbb{C}^d$ with $\mathbf{s}^{(1)}(\mathbf{z}) = Z$, $\mathbf{s}^{(1)}(\mathbf{w}) = W$, and $\langle \mathbf{z}, \mathbf{w} \rangle = 0$. Then we define, for $\mathcal{K} \subseteq \{i, j, +, -, 0\}^n$,

$$\mathcal{K}^* := \{W \in \{i, j, +, -, 0\}^n : W \perp Z \text{ for all } Z \in \mathcal{K}\}.$$

(This is analogous to the usual definition [BLSWZ, Sect. 3.4] for oriented matroids.)

**Example 5.1.** Here we compute the duals and double-duals of the 2-matroids of rank 1 listed in Figure 4.1.

(1) The complex matroid $\mathcal{K}_1 = \{00, ++, --, ii, jj\}$ is a complexified oriented matroid: $\mathcal{K}_1 = \mathcal{L} \circ i\mathcal{L}$ for $\mathcal{L} = \{00, ++, --\}$. We easily find $\mathcal{K}_1^* = \{00, +-, -+, ij, ji\} = \{00, +-, +-\}^* \circ i\{00, +-, +-\}^*$, which is again a complexified oriented matroid, and $\mathcal{K}_1^{**} = \mathcal{K}_1$.

(2) For the 2-matroid $\mathcal{K}_2 = \{00, +-, -+, ii, jj\}$ (which is quasi-complexified), but not complex, we find $\mathcal{K}_2^* = \{00, ii, jj\}$, which is not a 2-matroid.

(3) The dual of the complex matroid $\mathcal{K}_3 = \{00, +j, ij, i+, ii, -i, ji, j-, jj\}$ turns out to be a reorientation of it: $\mathcal{K}_3^* = \{00, j+, ji, +i, ii, i-, ij, -j, jj\} = {}_{-\{2\}}\mathcal{K}_3$, and thus $\mathcal{K}_3^{**} = \mathcal{K}_3$.

(4) For the 2-matroid $\mathcal{K}_4 = \{00, +i, ii, i+, ij, -j, jj, j-, ji\}$, finally we find $\mathcal{K}_4^* = \{00, ii, ij, jj, ji\}$, which again fails to be a 2-matroid.

This indicates that there probably is no reasonable duality theory for 2-matroids. However, for complex matroids a good duality theory can be expected. We will prove this here for the realizable case: whenever $\mathcal{K}$ is a realizable complex matroid, then $\mathcal{K}^*$ is also a complex matroid, and $\mathcal{K}^{**} = \mathcal{K}$. This amounts to a "complex Farkas Lemma"; we derive it from the following "real" transposition theorem 'of Farkas type'.

**Lemma 5.2.** [StW, p. 23] *Let $\mathbf{M} \in \mathbb{R}^{p \times n}$ and $\mathbf{N} \in \mathbb{R}^{q \times n}$ be real matrices. Then*
**either** *there is a vector $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{M}\mathbf{x} = \mathbf{0}$ and $\mathbf{N}\mathbf{x} > \mathbf{0}$ (component-wise),*
**or** *there are vectors $\mathbf{w} \in \mathbb{R}^p$, $\mathbf{u} \in \mathbb{R}^q$ with $\mathbf{M}^T\mathbf{w} + \mathbf{N}^T\mathbf{u} = \mathbf{0}$, $\mathbf{u} \geq \mathbf{0}$, $\mathbf{u} \neq \mathbf{0}$,*
*but not both.*

**Theorem 5.3.** *Let $V \subseteq \mathbb{C}^n$ be a complex subspace of dimension $r$, let $\mathcal{K} := \mathbf{s}^{(1)}(V)$ be the complex matroid it represents, and let $V^\perp \subseteq \mathbb{C}^n$ be the orthogonal complement of $V$. Then*

$$\mathcal{K}^* \;=\; \mathbf{s}^{(1)}(V^\perp).$$

*In particular, the dual $\mathcal{K}^*$ is a complex matroid, and $\mathcal{K}^{**} = \mathcal{K}$.*

**Proof.** First take $Z \in \mathbf{s}^{(1)}(V^\perp)$, with $Z = \mathbf{s}^{(1)}(\mathbf{z})$, and $W \in \mathcal{K}$, with $W = \mathbf{s}^{(1)}(\mathbf{w})$. Then we have $\mathbf{w}^T\overline{\mathbf{z}} = 0$ by construction, and thus $W \perp Z$. This proves $\mathbf{s}^{(1)}(V^\perp) \subseteq \mathcal{K}^*$.

For the reverse inclusion, assume that $Z \notin \mathbf{s}^{(1)}(V^\perp)$. After relabeling and reorientation, it suffices to consider $Z$ of the form

$$Z \;=\; (00\ldots00|{+}{+}\ldots{+}{+}|ii\ldots ii) \;\in\; \{0,+,i\}^n,$$

with $a$ zeroes, $b$ pluses and $c$ $i$'s, say.

Assuming $\dim_{\mathbb{C}}(V) = r$, we find $A \in \mathbb{C}^{r\times n}$ with $V^\perp = \{\mathbf{z} \in \mathbb{C}^n : A\mathbf{z} = \mathbf{0}\}$ and $V = \{\overline{A}^T\mathbf{w} : \mathbf{w} \in \mathbb{C}^r\}$.

Now $Z \notin \mathbf{s}^{(1)}(V^\perp)$ says that there is no $\mathbf{z} \in \mathbb{C}^n$ with $\mathbf{s}^{(1)}(\mathbf{z}) = Z$ and $A\mathbf{z} = 0$. Setting $\mathbf{z} = \mathbf{x}+i\mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $A = A_{\mathrm{Re}}+iA_{\mathrm{Im}}$ for $A_{\mathrm{Re}}, A_{\mathrm{Im}} \in \mathbb{R}^{r\times n}$, this means that there is no $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that

$$\begin{array}{ll}
A_{\mathrm{Re}}\mathbf{x} - A_{\mathrm{Im}}\mathbf{y} = \mathbf{0} & A_{\mathrm{Im}}\mathbf{x} + A_{\mathrm{Re}}\mathbf{y} = \mathbf{0} \\
(I_a \mid O \mid O)\mathbf{x} = \mathbf{0} & (I_a \mid O \mid O)\mathbf{y} = \mathbf{0} \\
(O \mid I_b \mid O)\mathbf{x} > \mathbf{0} & (O \mid I_b \mid O)\mathbf{y} = \mathbf{0} \\
 & (O \mid O \mid I_c)\mathbf{y} = \mathbf{0}
\end{array}$$

where $I_a$ denotes an identity matrix of size $a$, $O$ denotes a zero matrix and $\mathbf{0}$ is a zero vector of appropriate size. Equivalently, with $\widehat{\mathbf{x}} := \binom{\mathbf{x}}{\mathbf{y}}$:

$$\nexists\, \widehat{\mathbf{x}} \in \mathbb{R}^{2n} : \qquad \begin{array}{l}
\begin{pmatrix}
 & A_{\mathrm{Re}} & & & -A_{\mathrm{Im}} & \\
 & A_{\mathrm{Im}} & & & A_{\mathrm{Re}} & \\
I_a & O & O & O & O & O \\
O & O & O & I_a & O & O \\
O & O & O & O & I_b & O
\end{pmatrix} \widehat{\mathbf{x}} \;=\; \mathbf{0} \\
\begin{pmatrix}
O & I_b & O & O & O & O \\
O & O & O & O & O & I_c
\end{pmatrix} \widehat{\mathbf{x}} \;>\; \mathbf{0}
\end{array}$$

Thus, by Lemma 5.2 there exist $\widehat{\mathbf{w}} \in \mathbb{R}^{2r+n}$, $\widehat{\mathbf{u}} \in \mathbb{R}^{b+c}$ with $\widehat{\mathbf{u}} \geq \mathbf{0}$ and $\widehat{\mathbf{u}} \neq \mathbf{0}$ that satisfy

$$\begin{pmatrix}
 & & I_a & O & O \\
A_{\mathrm{Re}}^T & A_{\mathrm{Im}}^T & O & O & O \\
 & & O & O & O \\
 & & O & I_a & O \\
-A_{\mathrm{Im}}^T & A_{\mathrm{Re}}^T & O & O & I_b \\
 & & O & O & O
\end{pmatrix} \widehat{\mathbf{w}} \;+\; \begin{pmatrix}
O & O \\
I_b & O \\
O & O \\
O & O \\
O & O \\
O & I_c
\end{pmatrix} \widehat{\mathbf{u}} \;=\; \mathbf{0}.$$

Now putting $\widehat{\mathbf{w}} =: \begin{pmatrix} \mathbf{w}' \\ \mathbf{w}'' \\ \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{b}_2 \end{pmatrix}$ with $\mathbf{w} := \mathbf{w}' + i\mathbf{w}''$ and $\widehat{\mathbf{u}} =: \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{c}_2 \end{pmatrix}$, we can rewrite this as

$$\begin{aligned}
A_{\mathrm{Re}}^T\mathbf{w}' \;+\; A_{\mathrm{Im}}^T\mathbf{w}'' + \begin{pmatrix}\mathbf{a}_1\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix} \;+\; \begin{pmatrix}\mathbf{0}\\ \mathbf{b}_1\\ \mathbf{0}\end{pmatrix} \quad &= \quad \mathbf{0},\\
-A_{\mathrm{Im}}^T\mathbf{w}' \;+\; A_{\mathrm{Re}}^T\mathbf{w}'' + \begin{pmatrix}\mathbf{a}_2\\ \mathbf{0}\\ \mathbf{0}\end{pmatrix} \;+\; \begin{pmatrix}\mathbf{0}\\ \mathbf{b}_2\\ \mathbf{0}\end{pmatrix} \;+\; \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \mathbf{c}_2\end{pmatrix} \quad &= \quad \mathbf{0},
\end{aligned}$$

that is,

$$\overline{A}^T\mathbf{w} \;+\; \begin{pmatrix}\mathbf{a}_1 + i\mathbf{a}_2\\ \mathbf{b}_1 + i\mathbf{b}_2\\ i\mathbf{c}_2\end{pmatrix} \;=\; \mathbf{0} \qquad \text{for } \mathbf{b}_1, \mathbf{c}_2 \geq \mathbf{0}, \text{ not both equal to zero.}$$

Now letting $\tilde{\mathbf{w}} := i\overline{A}^T\mathbf{w} = \begin{pmatrix}\mathbf{a}_2 - i\mathbf{a}_1\\ \mathbf{b}_2 - i\mathbf{b}_1\\ \mathbf{c}_2\end{pmatrix}$, we find that there exists a vector $W = \mathbf{s}^{(1)}(\tilde{\mathbf{w}}) \in \mathcal{K}$ with

$$W \;=\; (\underbrace{**\ldots**}_{\text{arbitrary}}|\underbrace{**\ldots**}_{\in\{j,+,-,0\}}|\underbrace{\oplus\oplus\ldots\oplus\oplus}_{\in\{+,0\}})$$

with some $j$ in the middle part, or some $+$ in the third part, or both. But this implies that $W \not\perp Z$, and thus $Z \notin \mathcal{K}^*$. ◻

**Theorem 5.4.** *Let $\mathcal{K} = \mathcal{L}{\circ}i\mathcal{L}'$ be a quasi-complexified 2-matroid. Then $\mathcal{K}^* \subseteq \mathcal{L}^*{\circ}i\mathcal{L}^*$, and $\mathcal{K}^{**} \supseteq \mathcal{L}{\circ}i\mathcal{L}$.*

*In particular, the following three conditions are equivalent for quasi-complexified 2-matroids:*

*(i) $\mathcal{K}$ is a complex matroid,*

*(ii) $\mathcal{K}$ is a complexified oriented matroid,*

*(iii) $\mathcal{K}^* = \mathcal{L}^*{\circ}i\mathcal{L}^*$.*

**Proof.** To see $\mathcal{L}^*{\circ}i\mathcal{L}^* \subseteq \mathcal{K}^*$, assume that for some real $X', Y'$ we have $X'{\circ}iY' \perp X{\circ}iY$ for all $X{\circ}iY \in \mathcal{L}{\circ}i\mathcal{L}'$. In particular, this implies $X'{\circ}iY' \perp X$ for all $X \in \mathcal{L}$. Looking at the imaginary part, this yields $Y' \in \mathcal{L}^*$. Now assume that $Y'{\circ}X' \notin \mathcal{L}^*$, then there is some $X \in \mathcal{L}$ with $X \not\perp Y'{\circ}X'$. But if $\underline{X} \cap \underline{Y'} \neq \emptyset$, then we get $X \perp Y'{\circ}X'$ from $Y' \in \mathcal{L}^*$, and if $\underline{X} \cap \underline{Y'} = \emptyset$ but $\underline{X} \cap \underline{X'} \neq \emptyset$, then we get $X \not\perp X'{\circ}iY'$, a contradiction.

Next we verify that the other inclusion in the case $\mathcal{L} = \mathcal{L}'$. For this we take $X, Y \in \mathcal{L}$ and $X', Y' \in \mathcal{L}^*$ and have to show $X{\circ}iY \perp X'{\circ}iY'$. If $\underline{Y} \cap \underline{Y'} \neq \emptyset$, then this already implies the claim (for arbitrary real $X, X'$). If $\underline{Y} \cap \underline{Y'} = \emptyset$ but $\underline{Y} \cap \underline{X'} \neq \emptyset$, then we for corresponding vectors $\mathbf{y}$ and $\mathbf{x}'$ the sum $(i\mathbf{y})^T\overline{\mathbf{x}'}$ has summands of both positive and negative imaginary part, so we can choose $\mathbf{y}$ and $\mathbf{x}'$ to get a zero sum, which proves the claim. Thus we can assume $\underline{Y} \cap \underline{Y'} = \underline{Y} \cap \underline{X'} = \emptyset$, and by symmetry also $\underline{Y'} \cap \underline{X} = \emptyset$. But now orthogonality follows from $X \perp X'$.

¿From this we also get $\mathcal{K}^{**} \supseteq (\mathcal{L}^* \circ i\mathcal{L}^*)^* = \mathcal{L}^{**} \circ i\mathcal{L}^{**} = \mathcal{L} \circ i\mathcal{L}$.

Now (ii)$\Longrightarrow$(i) is obvious, and the converse is clear by looking at cocircuits, i.e. vectors with minimal non-empty supports. Also we have already verified (ii)$\Longrightarrow$(iii).

Now if (ii) fails, that is $\mathcal{L} \neq \mathcal{L}'$, then we can find $X \in \mathcal{L}$ and $Y \in (\mathcal{L}')^*$ with $X \not\perp Y$, since the underlying matroids coincide. This means that $iY \not\perp X$, and thus $X \notin \mathcal{K}^*$, and hence (iii) fails. $\square$

We note that last proof is most conveniently formulated in the language of fuzzy rings, where the scalar product $Z^T\overline{W} := \bigoplus_{e \in E} Z_e \odot \overline{W}_e$ can explicitly be defined and evaluated. We will discuss the set-up for this in the next section.

## 6. 2-matroids and 'matroids with coefficients'.

In this section we want to explain why the theory of "matroids with coefficients" by Dress & Wenzel [Dr1] [DW1,2,3] does not support complex matroids.

The key observation is the following. Consider the strata of the sign function $\mathbf{s}^{(1)}$, that is, the maximal subsets of the complex plane $\mathbb{C}$ that have the same complex sign. If we define the sum or product of two such strata pointwise, then we get that *the sum or product of two strata is again a union of strata.* This allows us to define a ring structure $K$ on the non-empty sets of signs, with zero element $\{0\}$, unit $\{+\}$, and negative unit $\{-\}$. In the following, we will simplify notation by omitting some set brackets.

**Lemma 6.1.** *The ring of non-empty sets of signs $K := 2^{\{i,j,+,-,0\}} \backslash \emptyset$ is a fuzzy ring in the sense of Dress [Dr1] with addition and multiplication defined element-wise by the following tables:*

| $\oplus$ | $0$ | $+$ | $-$ | $i$ | $j$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $+$ | $-$ | $i$ | $j$ |
| $+$ | $+$ | $+$ | $0{+}{-}$ | $i$ | $j$ |
| $-$ | $-$ | $0{+}{-}$ | $-$ | $i$ | $j$ |
| $i$ | $i$ | $i$ | $i$ | $i$ | $0{+}{-}ij$ |
| $j$ | $j$ | $j$ | $j$ | $0{+}{-}ij$ | $j$ |

| $\odot$ | $0$ | $+$ | $-$ | $i$ | $j$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $+$ | $0$ | $+$ | $-$ | $i$ | $j$ |
| $-$ | $0$ | $-$ | $+$ | $j$ | $i$ |
| $i$ | $0$ | $i$ | $j$ | $-ij$ | $+ij$ |
| $j$ | $0$ | $j$ | $i$ | $+ij$ | $-ij$ |

*A smaller subring of this ring $K$ is the fuzzy ring $K' = \{0, +, -, i, j, 0{+}{-}, {+}ij, {-}ij, 0{+}{-}ij\}$, which is generated by the one-element subsets of $\{i, j, +, -, 0\}$.*

**Proof.** The axioms are all trivial to check. We only note that in the notation of [Dr1] [DW1,2,3] for $K$ we have the zero element $0_K = \{0\}$, the one element $1_K = \{+\}$, the negative-one $\epsilon = \{-\}$, the subset $K_0 = \{S \subseteq \{i, j, +, -, 0\} : 0 \in S\}$ and the group of units $K^* = \{+, -\}$ (which is denoted $\dot{K}$ in [Dr1] and $K^\times$ in [DW1]). $K'$ is a fuzzy subring with the same zero and units. □

Now we can ask about the matroids with coefficients in $K$. It turns out, however, that $K$ has too few units (namely, only $+$ and $-$) to have non-trivial matroids with coefficients.

**Proposition 6.2.**

(i) *The matroids with coefficients in $K$ (or $K'$) on the ground set $[n]$ are canonically equivalent to the complexified oriented matroids on the same ground set.*

(ii) *Let $\mathcal{K} \subseteq \{i, j, +, -, 0\}^n$ be a 2-matroid. Then $\mathcal{K}$ presents a matroid with coefficients over $K$ (and over $K'$) if and only if it is a quasi-complexified oriented matroid, $\mathcal{K} = \mathcal{L}' \circ i\mathcal{L}$. Furthermore, $\mathcal{K}$ is in this case equivalent (as a matroid with coefficients in $K$ resp. $K'$) to the complexified oriented matroid $\mathcal{L}' \circ i\mathcal{L}'$.*

**Proof.** This is quite trivial once the definitions have been decoded. Since we do not want to go into details, we only sketch a few hints for the interested reader. The set of units of

$K$ and of $K'$ is $K^* = \{+,-\}$. The *support* of $Z \in K^n$ is $\underline{Z} = \{e \in [n] : Z_e \neq 0\}$, while the *proper support* is $\underline{\underline{Z}} := \{e \in [n] : Z_e \in \{+,-\}$.

(i) Let $\mathbf{K} \subseteq K^n$ be a matroid with coefficients in $K$ or $K'$. We only consider the *minimal presentation* of $\mathbf{K}$, which is a subset of $\{+,-,0\}^n$, using the defining property of a matroid with coefficients. Now it is easily seen that $\mathcal{C}^* \subseteq \{+,-,0\}^n$ is the minimal presentation of a matroid with coefficients if and only if it is the set of cocircuits $\mathcal{C}^* = \min(\mathcal{L}\backslash\mathbf{0})$ of an oriented matroid, and different oriented matroids define different matroids with coefficients.

(More generally, the matroids with coefficients over a fuzzy ring are canonically equivalent to the subring generated by 0 and the units; for $K$ and $K'$ this subring is $\{0,+,-,+-0\} = \mathbb{R}//\mathbb{R}^+$, the canonical fuzzy ring for oriented matroids.)

(ii) For a 2-matroid $\mathcal{K} \subseteq \{i,j,+,-,0\}^n$ to present a matroid with coefficients over $K$ it is required that for every $Z \in \mathcal{K}$ we find $Z' \in \mathcal{K} \cap \{+,-,0\}^n =: \mathcal{L}'$ with the same support. Thus $\mathcal{K} = \mathcal{L}' \circ i\mathcal{L}$ is quasi-complexified by condition (ii$'$) of Theorem 1.6(1). But this implies that $\mathcal{K}$, the oriented matroid $\mathcal{L}'$, and the complexified oriented matroid $\mathcal{L}' \circ i\mathcal{L}'$ present the same matroid with coefficients. $\square$

Thus the theory of matroids with coefficients does not cover complex matroids. The reason for this failure can be seen in the basic construction of the fuzzy rings that serve as coefficient domains: it is too closely modeled after a quotient of a ring by a subgroup of units. Thus it does not allow for the type of equivalence relations such as the finite topological stratifications that yield CW-models.

On the other hand the fuzzy rings $\mathbb{C}/\mathbb{R}^+$ and $\mathbb{C}//\mathbb{R}^+$, which the theory of matroids with coefficients would suggest for studying complex matroids, are infinite. Here $\mathbb{C}/\mathbb{R}^+$ is essentially the fuzzy ring of all subsets of the circle $S^1$, while $\mathbb{C}//\mathbb{R}^+$ is essentially the ring of all arcs on this circle.

However, a 'theory of matroids' with coefficients that allows for a topological structure is not available. (Such a theory would avoid the (interesting) Reorientation Problem 7.5.) This leads one to desire a theory of matroids with 'cellular' coefficient domains that would be more general (and simpler) than the Dress-Wenzel theory. This new theory should cover e.g. the present case of complex matroids and 2-matroids, and more general ones, like the $k$-matroids of Björner & Ziegler [BZ, Sect. 9.3].

## 7. Some problems.

In this section we want to, finally, sketch six areas of future research. All of them can be considered as major quality criteria for the concept of a 'complex matroid'. We note that in the context of matroids with coefficients, the duality is established in [Dr1], the basis version is available in [DW2], while the reduction to corank 2 is established (for perfect fuzzy rings) in [DW3]. We hope to attack these problems in future work.

**7.1 Duality.** We assume that the proof of Theorem 5.3 can be 'lifted' to an oriented matroid setting, to prove that $\mathcal{K}^*$ is a complex matroid and $\mathcal{K}^{**} = \mathcal{K}$ also in the non-representable case. This certainly seems plausible, since it also works for a complexified oriented matroid (Theorem 5.4).

The alternative would be to construct $\mathcal{K}^*$ from an oriented matroid frame of $\mathcal{K}$. The problem here is that it is not clear that the !-construction behaves well under duality. However, from the well-known formula for the rank function of the dual matroid it is easy to see that if $\mathcal{L}!$ defines a real frame for a 2-matroid, then the dual oriented matroid is the real frame for a 2-matroid as well. The goal would be to adjust the 'free parameters' of the !-construction in such a way that the 2-matroid (which can be defined in terms of $\mathcal{L}!^*$) coincides with $\mathcal{K}^*$.

**7.2 Basis axioms.** Assume that a complex hyperplane arrangement in $\mathbb{C}^d$ is given as $\mathcal{B} = \{\ker(\ell_1), \ldots, \ker(\ell_n)\}$. Then all the relevant data are encoded in the set of *Grassmann-Plücker maps*

$$\chi : \binom{[n]}{d} \longrightarrow \{i, j, +, -, 0\}, \qquad \{a_1, \ldots, a_d\}_< \longmapsto \mathbf{s}^{(1)} \left( \alpha \ \det{}_{\mathbb{C}} \left(\ell_{a_1}, \ldots, \ell_{a_d}\right)\right)$$

that we get for various $\alpha \in \mathbb{C}$. The problem is to characterize 2-matroids and complex matroids in terms of their Grassmann-Plücker maps.

**7.3 Corank 2.** It is one of the deeper properties of oriented matroids that important structural properties can be reduced to corank 2 (see [LV] [BLSWZ, Sects. 3.6, 7.1]). Similar reductions should be possible for 2-matroids and for complex matroids. Conjecture 4.8 is an attempt in this direction. In the basis version, one would ask for a reduction of the general Grassmann-Plücker conditions on a 2-matroid to the 3-term conditions.

We note here that this type of problem becomes non-trivial because a complex matroid is not in general determined by its cocircuit signatures. This means that, for example, it is not good enough to rely on signed cocircuits when constructing the dual complex matroid. Also, there does not seem to be a simple complex analogue for the conformal decomposition of covectors into cocircuits, which is an extremely useful tool for similar situations in the oriented matroid case [BLSWZ, Prop. 3.7.2].

**7.4 Orientability.** The following problem for 2-matroids and complex matroids is nearly untouched: which matroids $M$ occur as the underlying matroids of 2-matroids resp. complex matroids? In other words, which matroids are 2-*orientable* resp. *complex orientable*?

In particular, one should give examples (with proof!) of matroids that are

— not 2-orientable

— 2-orientable but not complex orientable

— complex orientable but not complex realizable.

We have obvious candidates at least for the first two cases, namely the Fano matroid $F_7$ resp. the non-Desargues matroid, but no effective criteria for non-orientability. The fact that the non-Desargues matroid is 2-realizable [GoM, p. 257] shows that non-canonical things (from the point of view of matroid representation theory) can happen.

**7.5 Complex $\mathbf{s}^{(2)}$-matroids.** Let $\mathcal{B} = \{H_1, \ldots, H_n\}$ be a linear complex hyperplane arrangement in $\mathbb{C}^d$, and consider the $\mathbf{s}^{(2)}$-stratification $\mathbf{s}^{(2)}_{\mathcal{B}} : \mathbb{C}^d \longrightarrow \{+,-,0\}^{2n}$ that we get from

$$\left(\mathbf{s}^{(2)}_{\mathcal{B}}(\mathbf{z})\right)_a = \begin{cases} \mathbf{s}(\Im(\ell_a(\mathbf{z}))) & \text{if } 1 \le a \le n, \\ \mathbf{s}(\Re(\ell_{a-n}(\mathbf{z}))) & \text{if } n < a \le 2n. \end{cases}$$

This yields an oriented matroid frame $\mathcal{K}^{(2)} \subseteq \{+,-,0\}^{2n}$ for the 2-matroid $\mathcal{K}^{(1)}(\mathcal{B})$. Furthermore, with the particular choice of taking real and complex parts we get that $\mathcal{K}^{(2)}$ admits an action of the dihedral group $I_2(4)$ (the symmetry groups of the square), induced by multiplication with '$+i$' and conjugation, which permutes and reorients the elements of $\mathcal{K}^{(2)}$.

One is lead to ask [Dr2] whether this $I_2(4)$-action can be used to define a *complex structure* for even $\mathbf{s}^{(2)}$-oriented matroids $\mathcal{K}^{(2)} \subseteq \{+,-,0\}^{2n}$. In particular, would every complex matroid $\mathcal{K}^{(1)} \subseteq \{i,j,+,-,0\}^n$ admit a '$\mathcal{K}^{(2)}$-frame' of this symmetric type? Conversely, is the 2-matroid derived from such a $\mathcal{K}^{(2)}$-matroid with $I_2(4)$-action complex?

**7.6 Reorientation.** The problem of characterizing which complex matroids realize the same (i.e., homeomorphic) 2-arrangements in $S^{2r-1}$ is subtle. We note that this *reorientation problem* corresponds to rescaling operations $\ell_a \longrightarrow \alpha_a \ell_a$ ($\alpha_a \in \mathbb{C}\backslash 0$) in the realized case.

Clearly, if $\mathcal{L}! \cup f$ is an extension of $\mathcal{L}!$ such that $\{e, e^!, f\}$ forms a 3-circuit, then the corresponding exchange $\mathcal{K} \longrightarrow (\mathcal{K} \cup f)\backslash e$ preserves the 2-arrangement. However, it is not at all clear that exchanges of this type connect every pair of complex matroids that represent the same 2-arrangement.

In fact, we note that the oriented matroid program $(\mathcal{L}!, e, e^!)$ can be *non-euclidean* in the sense of Edmonds & Mandel [EdM] [BLSWZ, Sect. 10.5], so the space of such extensions can be disconnected [StZ]. The smallest examples for this occur over $U_{2,4}$, corresponding to non-euclidean orientations $\mathcal{L}!$ of $U_{4,8}$. This indicates that the *reorientation space* of a complex matroid is not a torus in general.

**7.7 Affine complex matroids.** If $\mathcal{K} \subseteq \{i, j, +, -, 0\}^n$ is a complex matroid and $g \in [n]$ is a fixed element (not a loop), then the sign vector system $\mathcal{K}^+ := \{Z \in \mathcal{K} : Z_g = +\}$ is an *affine complex matroid.* Note that $0 \cup (\mathcal{K}^+ \cap \mathcal{K}_{comp})$ is again the face poset of a regular $CW$ complex $\Gamma^+_{comp}$.

If $\mathcal{K}$ is complex realizable, then then $\mathcal{K}^+$ is the sign vector system of an affine complex hyperplane arrangement, and $\Gamma^+_{comp}$ has the homotopy type of the complement of this affine arrangement; in particular, we get

$$\Gamma_{comp} \;\simeq\; \Gamma^+_{comp} \times \mathbb{C}^*. \tag{$*$}$$

The problem is to prove $(*)$ also in the case where $\mathcal{K}$ is not realizable.

In the case of a complexified oriented matroid, we have (in the notation of [BLSWZ, Sect. 4.1])

$$\mathcal{K}^+ \;=\; (\mathcal{L} \circ i\mathcal{L})^+ \;=\; \{X \circ iY : X_g = +,\ Y_g = 0\} \;\cong\; \mathcal{L}^+ \circ i(\mathcal{L}/g),$$

and $\mathcal{K}^+ \cap \mathcal{K}_{comp} = \mathcal{T} \circ i(\mathcal{L}/g)$ is the affine *Salvetti complex* [Sal] of an oriented matroid, where the linear case was analyzed in [BZ, Sect. 4].

Furthermore, one wants to characterize the affine systems $\mathcal{K}^+$ intrinsically. The recent work of Karlander [Kar] in the oriented matroid case suggests that this might not be easy.

**7.8 Spaces of complex matroids.** Using complex matroids, one can try to define *complex combinatorial differential manifolds* in analogy of MacPherson's real combinatorial differential manifolds [MPh].

The classifying space for complex vector bundles over such manifolds should be the *CM-Grassmannian*: the order complex of the set of all complex matroids of rank $r$ on the ground set $[n]$, ordered by *weak maps* $\mathcal{K}^1 \rightsquigarrow \mathcal{K}^2$. These are defined by the condition that for every $Z \in \mathcal{K}^2$ there is a $W \in \mathcal{K}^1$ with $W \geq Z$ — see [BLSWZ, Prop. 7.7.5] for the oriented matroid case.

We refer to [GeM], [Bab], [MPh] and [MnZ] for recent work in this direction for the real case.

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