A New Local Criterion for the Lattice Property

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Abstract.
We prove that a bounded poset of finite length is a lattice if and only if the following condition holds: whenever two elements x_1, x_2 that cover a common element x are both smaller that two elements y_1, y_2 that are covered by a common element y, then there exists an element z that is an upper bound for x_1, x_2 and a lower bound for y_1, y_2.

For posets that arise in various combinatorial and geometric situations it is nontrivial to check whether they are lattices. In [2] it was observed that for this it suffices to check a local condition: the existence of a join for any pair of elements that cover a common element. In this note we give a new local criterion for the lattice property: it only considers two covers of an element x, which are both smaller than two cocovers of an element y, and requires that there is an element z between the two covers of x and the two cocovers of y. Furthermore, our new criterion is self-dual (that is, equivalent to its order dual formulation), and it does not any more presuppose the existence of lattice operations for any pair of elements. It was used in [4] to study the lattice property for higher Bruhat orders.

For background on elementary properties of posets see [1] and [3, Chap. 3]. We need the following concepts and terminology. All posets P considered in this note are *bounded*: they have a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. A *chain* in a poset is a totally ordered subset, its *length* is the cardinality minus 1. The length $\ell(P)$ of a poset is the maximal length of a chain in P. The posets we consider need not be finite, but they have finite length. The closed interval $\{z \in P : x \leq z \leq y\}$ is denoted by $[x, y]$. We say that y *covers* x if $x < y$ and there is no z with $x < z < y$, so $[x, y] = \{x, y\}$. In this situation, denoted by $x \lessdot y$, we say that y is a *cover* of x, and x is a *cocover* of y. An *atom* is a cover of $\hat{0}$, while a *coatom* is a cocover of $\hat{1}$. Note that if x is an atom, then $\ell[x, \hat{1}] < \ell(P)$.

An *upper bound* of x_1 and x_2 is an element y with $x_1 \leq y$ and $x_2 \leq y$. If two elements $x_1, x_2 \in P$ have an upper bound y_0 so that $y_0 \leq y$ for every other upper bound y, then this element y_0 is called the *join* of x_1 and x_2 and denoted by $y_0 = x_1 \vee x_2$. In a poset of finite length, this is equivalent to the requirement that y_0 is the unique minimal upper bound of

x_1 and x_2. Similarly, the *meet* of x_1 and x_2 is their unique maximal lower bound, denoted by $x_1 \wedge x_2$, if this exists. A poset P is a *lattice* if the meet and the join exists for every pair $x_1, x_2 \in P$. It is easy to see that in posets of finite length it is sufficient to require that joins exist, or that meets exist.

The following lemma shows that if the join $x_1 \vee x_2$ does not exist, then there exist upper bounds with a common cover that serve as a certificate. As an immediate corollary we get a criterion that was designed in [2] to check the lattice property for "posets of regions". For the following, we use expressions like "$x \lessdot x_1, x_2$" to denote that $x \lessdot x_1$ and $x \lessdot x_2$.

Lemma. *Let P be a bounded poset of finite length, and $X_1, X_2 \in P$. Then $X_1 \vee X_2$ exists if and only if there are no elements $y_1, y_2 \lessdot y$ with a common cover in P which are both upper bounds for X_1 and X_2, for which there is no element Z with $X_i \leq Z \leq y_j$ for $i, j \in \{1, 2\}$.*

Proof. If $X_1 \vee X_2$ exists, then we can always put $Z := X_1 \vee X_2$. For the converse we use induction on the length of P. For $\ell(P) \leq 2$ the claim is trivial since P is automatically a lattice. Let $Y_1, Y_2 \in P$ be minimal upper bounds for X_1, X_2. If $Y_1 = \hat{1}$ or $Y_2 = \hat{1}$ then we get $Y_1 = Y_2$ from minimality. Otherwise we can choose coatoms y_1, y_2 so that $Y_1 \leq y_1 \lessdot \hat{1}$ and $Y_2 \leq y_2 \lessdot \hat{1}$. Then by assumption there is an element $Z \in P$ with $X_1, X_2 \leq Z \leq y_1, y_2$. Now Z and Y_1 are both upper bounds for X_1 and X_2 in the interval $[\hat{0}, y_1]$ of P, where $X_1 \vee X_2$ exists by induction on length. Since Y_1 is a minimal upper bound, this implies $Y_1 \leq Z$. Analogously, we get $Y_2 \leq Z$. This means that Y_1, Y_2 are minimal upper bounds for X_1 and X_2 in the interval $[\hat{0}, Z]$, where $X_1 \vee X_2$ exists by induction. This implies $Y_1 = Y_2 = X_1 \vee X_2$. □

Criterion 1. [2, Lemma 2.1] *Let P be a bounded poset of finite length. P is a lattice if and only if the following condition holds: whenever $x_1, x_2 \in P$ cover a common element x, they have a join $z = x_1 \vee x_2$.*

Proof. It is sufficient to show that $Y_1 \wedge Y_2$ exists for all Y_1, Y_2. Consider two lower bounds $x_1, x_2 \gtrdot x$. By the lemma (dualized), the existence of $Z := x_1 \vee x_2$ implies that $Y_1 \wedge Y_2$ exists. □

To apply Criterion 1, one has to still have control over the existence of joins for pairs of elements $x_1, x_2 \gtrdot x$ with a common cocover. However, the lemma above provides again a local obstruction for this, which leads to the following.

Criterion 2. *Let P be a bounded poset of finite length. P is a lattice if and only if the following condition holds: whenever there are covers x_1, x_2 of x and cocovers y_1, y_2 of y such that both x_1 and x_2 are smaller than both y_1 and y_2, there exists $z \in P$ that is an upper bound for x_1, x_2 and a lower bound for y_1, y_2.*

In symbols, P is a lattice if and only if $x \lessdot x_1, x_2 \leq y_1, y_2 \lessdot y$ implies the existence of $z \in P$ with $x \lessdot x_1, x_2 \leq z \leq y_1, y_2 \lessdot y$.

Proof. The condition is necessary: if P is a lattice, then $x \lessdot x_1, x_2 \leq x_1 \vee x_2 \leq y_1 \vee y_2 \leq y_1, y_2 \lessdot y$ and hence we can choose $z \in [x_1 \vee x_2, y_1 \vee y_2]$.

For sufficiency we use Criterion 1: it suffices to show that $x_1 \vee x_2$ exists for $x_1, x_2 \gtrdot x$. If $x_1 \vee x_2$ does not exist, then we get the required elements $y_1, y_2 \lessdot y$ from the lemma. $\square$

Both criteria yield straightforward algorithms to test the lattice property. For this we need a data structure for the poset P that provides lists of the covers and cocovers of any element: this amounts to storing lists of predecessors an successors for the directed graph of the Hasse diagram. Then a spanning tree algorithm can be used to identify the elements that are larger than x_1, x_2 and smaller than y_1, y_2. We omit details of a complexity analysis.

Finally we observe that the assumption of finite length is essential for the above criteria, even if we impose the condition $(*)$ that whenever $X < Y$ is not a cover relation, then there exist x, y with $X \lessdot x \leq y \lessdot Y$. This can be seen as follows. Let P be any bounded poset. The *ordinal product* $P \otimes \mathbb{Z}$ is the partial order on $P \times \mathbb{Z}$ where $(x, n) \leq (y, m)$ if either $x = y$ and $n \leq m$, or $x < y$, see [3, p. 101]. Let Q denote the interval $[(\hat{0}, 0), (\hat{1}, 0)]$ of $P \otimes \mathbb{Z}$. Then Q is a bounded poset in which every element except for $\hat{1}$ has a unique cover, and every element except for $\hat{0}$ has a unique cocover. Furthermore the poset Q satisfies $(*)$. However, Q is not a lattice unless P is a chain: if x_1, x_2 are incomparable elements of P, then $(x_1, 0)$ and $(x_2, 0)$ do not have a join — if (z, n) is an upper bound, then $(z, n-1)$ is a smaller upper bound.

References

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