On the Height of the Minimal Hilbert Basis

Jiyong Liu, Leslie E. Trotter, Jr. & Günter M. Ziegler

Abstract. We present an elementary proof of a result due to Ewald and Wessels: in a pointed, polyhedral cone of dimension $n \geq 3$ with integer-valued generators, any linearly independent generator representation for a minimal Hilbert basis element has coefficient sum less than $n-1$. Our proof makes explicit use of the geometry of the polyhedron given by the convex hull of the Hilbert basis elements.

We consider here subsets of $\mathbb{R}^n$ of the form $K = \{\sum_{i=1}^m \lambda_i a_i : \lambda_i \geq 0, 1 \leq i \leq m\}$, with $a_1, \ldots, a_m \in \mathbb{Q}^n$, i.e., finitely generated, rational *(convex) cones.* We denote $K = \mathit{cone}\{a_1, \ldots, a_m\}$. K is *nontrivial* if it properly contains $\{0\}$ and *pointed* if it contains no linear subspace properly containing $\{0\}$. When K is nontrivial and pointed, its *extreme rays* provide a unique (up to positive scaling) minimal set of generators.

When K is finitely generated, there also exists a finite set of vectors, a *Hilbert basis*, which generates $K \cap \mathbb{Z}^n$ under nonnegative integral combinations. For instance, when $K = \mathit{cone}\{a_1, \ldots, a_m\}$, with $a_1, \ldots, a_m$ integral, the integral elements in the (half-open) *zonotope* $\{\sum_{i=1}^m \lambda_i a_i : 0 \leq \lambda_i < 1, 1 \leq i \leq m\}$ constitute a Hilbert basis for K. When K is nontrivial and pointed, then its integral elements have a unique *minimal* Hilbert basis consisting of those members of $K \cap \mathbb{Z}^n$ which are not sums of other members of $K \cap \mathbb{Z}^n$. Our interest here is in properties of minimal Hilbert bases and we thus assume henceforth that $K = \mathit{cone}\{a_1, \ldots, a_m\}$ is nontrivial and pointed. We also assume, for $1 \leq i \leq m$, that a_i is the *shortest integral* vector on the extreme ray $\{\lambda a_i : \lambda \geq 0\}$, i.e., that $a_i \in \mathbb{Z}^n$ and $\mathit{gcd}\{a_{ij} : 1 \leq j \leq n\} = 1$, where a_{ij} denotes the j-th component of the vector a_i. Finally, we assume with no loss in generality that K is *full-dimensional*; i.e., $\mathit{rank}\{a_1, \ldots, a_m\} = n$. For background material on cones and Hilbert bases, the reader is referred to Schrijver [5].

Now let x be an element of the minimal Hilbert basis for K. By Carathéodory's Theorem, there are linearly independent generators $a_{i_1}, \ldots, a_{i_n}$ so that $x \in \mathit{cone}\{a_{i_1}, \ldots, a_{i_n}\}$;

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i.e., x is in the *simplicial* subcone of K generated by $a_{i_1}, \ldots, a_{i_n}$. Of course, x may be contained in many such simplicial subcones, and we define the *height* of x as

$$h(x) \;=\; \max\{\sum_{i=1}^{n} \lambda_i \;:\; x = \lambda_1 a_{i_1} + \ldots + \lambda_n a_{i_n};\; \lambda_i \geq 0,\; 1 \leq i \leq n;\; rank\{a_{i_1}, \ldots, a_{i_n}\} = n\}.$$

Motivated by a question of algebraic geometry, G. Ewald [1] posed the problem of determining an upper bound on $h(x)$. Subsequently, Ewald and Wessels [2] established that $h(x) \leq n-1$ and demonstrated, moreover, that this bound is sharp. Here we indicate an elementary proof of this result, discovered independently and motivated by results in Liu [3]. We also see that equality $h(x) = n-1$ cannot occur for $n > 2$. Note how the proof is facilitated by explicit use of the geometry of the polyhedron given by the convex hull of the Hilbert basis elements.

Theorem. *Let K be an n-dimensional pointed polyhedral rational cone, $n \geq 3$, and let x be a minimal Hilbert basis element for K. Then $h(x) < n-1$, and this bound is best-possible.*

Proof. Suppose $x \in C'$, where $C' = cone\{a_1, \ldots, a_n\}$ and the generators $a_1, \ldots, a_n$ define a simplicial subcone of K. We consider the polyhedron defined by the nonzero, integral elements of C'; i.e.,

$$P' \;:=\; conv\{\mathbb{Z}^n \cap (C' \setminus \{0\})\}.$$

The facets of P' separating P' from the origin induce a partition of C' into subcones, each generated by elements of the minimal Hilbert basis for C'. If any of these facets has an integral element in its (relative) interior, this element can be used to further refine the partition. Moreover, by triangulating, we may assume that each cone in the partition is simplicial (see [4, Theorem 3.4] for details). We will thus assume that C' is partitioned into simplicial subcones, say $C' = \bigcup_{i=1}^{p} C_i$, with each C_i generated by elements of the minimal Hilbert basis of C'. Consider $x \in C_1 = cone\{b_1, \ldots, b_n\}$, where $b_1, \ldots, b_n$ define a facet of P', where b_i is in the minimal Hilbert basis for C', with $b_i = \sum_{j=1}^{n} \lambda_{ij} a_j$, $\sum_{j=1}^{n} \lambda_{ij} \leq 1$ and $\lambda_{ij} \geq 0$, for $1 \leq i, j \leq n$.

Now x is in the minimal Hilbert basis for K, hence also for the subcones C' and C_1. Thus x is in the parallelotope defined by $b_1, \ldots, b_n$ and we can write $x = \sum_{i=1}^{n} \alpha_i b_i$, where $0 \leq \alpha_i \leq 1$, for $1 \leq i \leq n$. If $\sum_{i=1}^{n} \alpha_i > n-1$, then the element x' obtained by reflecting x from the apex, $\sum_{i=1}^{n} b_i$, of the parallelotope, i.e., $x' = (\sum_{i=1}^{n} b_i) - x = \sum_{i=1}^{n} (1-\alpha_i) b_i$, would be an integral vector of C_1, and hence of C', not in P'. Thus we must have $\sum_{i=1}^{n} \alpha_i \leq n-1$. Finally, since $x = \sum_{i=1}^{n} \alpha_i b_i = \sum_{i=1}^{n} \alpha_i (\sum_{j=1}^{n} \lambda_{ij} a_j) = \sum_{j=1}^{n} (\sum_{i=1}^{n} \alpha_i \lambda_{ij}) a_j$, with $\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \lambda_{ij} \leq n-1$, it follows that the height of x is bounded above by $n-1$.

Furthermore, if $\sum_{i=1}^{n} \alpha_i = n-1$, then the reflected point x' has to coincide with one of the vectors b_j, by construction. But this implies that $x = \sum_{i \neq j} b_i$, which is not an element of the minimal Hilbert basis for $n \geq 3$.

To see that the bound is sharp, let $d > 1$ be an integer and suppose $K \subseteq \mathbb{Q}^n$ is the simplicial cone generated by $(1, \ldots, 1, d)$ and the $n-1$ unit vectors $e_1, \ldots, e_{n-1}$. It is not difficult to see that the minimal Hilbert basis for K consists of the d vectors $\frac{t}{d}(1,\ldots,1,d)+\sum_{i=1}^{n-1}\frac{d-t}{d}e_i$, for $0\le t\le d-1$, along with the n generators of K. The vector of maximum height is achieved for $t=1$, i.e., for $x=\frac{1}{d}(1,\ldots,1,d)+\sum_{i=1}^{n-1}\frac{d-1}{d}e_i$. Thus $h(x)=\frac{1}{d}+\frac{(n-1)(d-1)}{d}=(n-1)-\frac{n-2}{d}$, so that as d increases, $h(x)$ is arbitrarily close to $n-1$. ☐

The class of examples used in the proof to demonstrate that the height bound is as tight as possible was also discovered by Ewald and Wessels [2]. Note that for these examples, the nonzero vectors in the parallelotope for K coincide with the minimal Hilbert basis. Such *parallelotope cones* are studied further in Liu [3]. In particular, it is shown for any parallelotope cone that each element of the minimal Hilbert basis which is also distinct from the generators of the cone must have height at least $1+\frac{n-2}{d}$, where d is the determinant of the $n\times n$ matrix formed by the cone's generators. Furthermore, for $n=3$ this condition characterizes parallelotope cones (see [4, Theorems 4.2 and 4.3]).

Finally, we remark that the proof here still applies for any minimal Hilbert basis (not necessarily unique) when K is not pointed; in this situation the height of any element of any minimal Hilbert basis is less than $n-1$.

References

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Jiyong Liu
School of OR&IE, Cornell University, Ithaca, New York 14853, USA.
Email: jiyong@cs.cornell.edu

Leslie E. Trotter, Jr.
School of OR&IE, Cornell University, Ithaca, New York 14853, USA, and
Département de Mathématiques, E.P.F.L., CH-1015 Lausanne-Ecublens, Switzerland
Email: ltrotter@orie.cornell.edu, trotter@eldi.epfl.ch

Günter M. Ziegler
ZIB-Berlin, Heilbronner Str. 10, W-1000 Berlin 31, Germany.
Email: ziegler@zib-berlin.de

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